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Hidden Symmetries and Dirac Fermions

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Abstract

In this paper, two things are done. First, we analyze the compatibility of Dirac fermions with the hidden duality symmetries which appear in the toroidal compactification of gravitational theories down to three spacetime dimensions. We show that the Pauli couplings to the p-forms can be adjusted, for all simple (split) groups, so that the fermions transform in a representation of the maximal compact subgroup of the duality group G in three dimensions. Second, we investigate how the Dirac fermions fit in the conjectured hidden overextended symmetry G^{++}. We show compatibility with this symmetry up to the same level as in the pure bosonic case. We also investigate the BKL behaviour of the Einstein-Dirac-p-form systems and provide a group theoretical interpretation of the Belinskii-Khalatnikov result that the Dirac field removes chaos.

1 Introduction

The emergence of unexpected ("hidden") symmetries in the toroidal dimensional reduction of gravitational theories to lower dimensions is a fascinating discovery whose full implications are to a large extent still mysterious [1, 2]. This result, which underlies U-duality [3], appears quite clearly when one dimensionally reduces down to 3 spacetime dimensions since the Lagrangian can then be rewritten as the coupled Einstein-scalar Lagrangian, where the scalars parametrize the symmetric space $G/K(G)$. Here, G is the "hidden" symmetry group and $K(G)$ its maximal compact subgroup (see [4] for a systematic study as well as the earlier work [5]). It has been argued that this G-symmetry signals a much bigger, infinite-dimensional, symmetry, which would be the overextension G^{++} [6, 7, 8, 9, 10], or even the very extended extension G^{+++}, of G [11, 12] (or perhaps a Borcherds superalgebra related to it [13]).

One intriguing feature of the hidden symmetries is the fact that when the coupled Einstein-$G/K(G)$ system is the bosonic sector of a supergravity theory, then, important properties of supergravities which are usually derived on the grounds of supersymmetry may alternatively be obtained by invoking the hidden symmetries. This is for instance the case of the Chern-Simons term and of the precise value of its coefficient in eleven dimensional supergravity, which is required by supersymmetry [14], but which also follows from the E (E_8 or E_{10}) symmetry of the Lagrangian [2, 10]. Quite generally, the spacetime dimension 11 is quite special for the Einstein-3-form system, both from the point of view of supersymmetry and from the existence of

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hidden symmetries. Another example will be provided below (subsection 5.2). One might thus be inclined to think that there is a deep connection between hidden symmetries and supergravity. However, hidden symmetries exist even for bosonic theories that are not the bosonic sectors of supersymmetric theories. For this reason, they appear to have a wider scope.

In order to further elucidate hidden symmetries, we have investigated how fermions enter the picture. Although the supersymmetric case is most likely ultimately the most interesting, we have considered here only spin-1/2 fermions, for two reasons. First, this case is technically simpler. Second, in the light of the above comments, we want to deepen the understanding of the connection — or the absence of connection — between hidden symmetries and supersymmetry.

The Einstein-(p-form)-Dirac system by itself is not supersymmetric and yet we find that the Dirac fermions are compatible with the G-symmetry, for all (split) real simple Lie groups. Indeed, one may arrange for the fermions to form representations of the compact subgroup $K(G)$. This is automatic for the pure Einstein-Dirac system. When p-forms are present, the hidden symmetry invariance requirement fixes the Pauli couplings of the Dirac fermions with the p-forms, a feature familiar from supersymmetry. In particular, E_8-invariance of the coupling of a Dirac fermion to the Einstein-3-form system reproduces the supersymmetric covariant Dirac operator of 11-dimensional supergravity [14, 15]. A similar feature holds for $\mathcal{N}=1$ supergravity in 5 dimensions [16]. Thus, we see again that hidden symmetries of gravitational theories appear to have a wider scope than supersymmetry but yet, have the puzzling feature of predicting similar structures when supersymmetry is available.

We formulate the theories both in 3 spacetime dimensions, where the symmetries are manifest, and in the maximum oxidation dimension, where the Lagrangian is simpler. To a large extent, one may thus view our paper as an extension of the oxidation analysis [6, 17, 4, 9, 13, 18, 19] to include Dirac fermions. Indeed the symmetric Lagrangian is known in 3 dimensions and one may ask the question of how high it oxidizes. It turns out that in most cases, the Dirac fermions do not bring new obstructions to oxidation in addition to the ones found in the bosonic sectors. If the bosonic Lagrangian lifts up to n dimensions, then the coupled bosonic-Dirac Lagrangian (with the Dirac fields transforming in appropriate representations of the maximal compact subgroup $K(G)$) also lifts up to n dimensions. This absence of new obstructions coming from the fermions is in line with the results of Keurentjes [20], who has shown that the topology of the compact subgroup $K(G)$ is always appropriate to allow for fermions in higher dimensions when the bosonic sector can be oxidized.

We then investigate how the fermions fit in the conjectured infinite-dimensional symmetry G^{++} and find indications that the fermions form representations of $K(G^{++}))$ up to the level where the matching works for the bosonic sector. We study next the BKL limit [21, 22] of the systems with fermions. We extend to all dimensions the results of [23], where it was found that the inclusion of Dirac spinors (with a non-vanishing expectation value for fermionic currents) eliminates chaos in four dimensions. Our analysis provides furthermore a group theoretical interpretation of this result: elimination of chaos follows from the fact that the geodesic motion on the symmetric space $G^{++}/K(G^{++})$, which is lightlike in the pure bosonic case [22], becomes timelike when spin-1/2 fields are included (and their currents acquire non-vanishing values) — the mass term being given by the Casimir of the maximal compact subgroup $K(G^{++})$ in the fermionic representation.

Our paper is organized as follows. In the next section, we collect conventions and notations. We then consider the dimensional reduction to three dimensions of the pure Einstein-Dirac system in D spacetime dimensions and show that the fermions transform in the spinorial representation of the maximal compact subgroup $SO(n+1)$ in three dimensions, as they should (section 3). The $SO(n,n)$ case is treated next, by relying on the pure gravitational case. The maximal compact subgroup is now $SO(n) \times SO(n)$. We show that one can choose the Pauli couplings so that the fermions transform in a representation of $SO(n) \times SO(n)$ (in fact, one can adjust the

Pauli couplings so that different representations arise) (section 4). In section 5, we turn to the E_n-family. We show that again, the Pauli couplings can be adjusted so that the spin-1/2 fields transform in a representation of $SO(16)$, $SU(8)$ or $Sp(4)$ when one oxidizes the theory along the standard lines. We then show that the G_2-case admits also covariant fermions in 5 dimensions (section 6) and treat next all the other non simply laced groups from their embeddings in simply laced ones (section 7).

In section 8, we show that the Dirac fields fit (up to the same level as the bosons) into the conjectured G^{++} symmetry, by considering the coupling of Dirac fermions to the (1+0) non linear sigma model of [10]. In section 9, we analyze the BKL limit and argue that chaos is eliminated by the Dirac field because the Casimir of the $K(G^{++})$ currents in the fermionic representation provides a mass term for the geodesic motion on the symmetric space $G^{++}/K(G^{++})$. Finally, we close our paper with some conclusions and technical appendices.

In the analysis of the models, we rely very much on the papers [24, 25, 4], where the maximally oxidized theories have been worked out in detail and where the patterns of dimensional reduction that we shall need have been established.

2 Conventions

2.1 Chevalley-Serre presentation and Cartan-Chevalley involution

We adopt the standard Chevalley-Serre presentation of the Lie algebras in terms of $3r$ generators $\{h_i, e_i, f_i\}$ ($i = 1, \cdots, r$ with r equal to the rank) as given for instance in [26], except that we take the “negative" generators f_i with the opposite sign (with respect to [26]). The only relation that is modified is $[e_i, f_j] = -\delta_{ij}\, h_i$. The other relations are unchanged, namely, $[h_i, h_j] = 0$, $[h_i, e_j] = A_{ij} e_j$, $[h_i, f_j] = -A_{ij} f_j$ (where A_{ij} is the Cartan matrix) and $ad_{e_i}^{1-A_{ij}} e_j = 0$, $ad_{f_i}^{1-A_{ij}} f_j = 0$.

The sign convention for f_i simplifies somewhat the form of the generators of the maximal compact subalgebra. The Cartan-Chevalley involution reads $\tau(h_i) = -h_i$, $\tau(e_i) = f_i$, $\tau(f_i) = e_i$ and extends to the higher height root vectors as $\tau(e_\alpha) = f_\alpha$, $\tau(f_\alpha) = e_\alpha$ where the e_α's are given by multi-commutators of the e_i's and the f_α's are given by multi-commutators of the f_i's in the same order (e.g., if $e_{r+1} = [e_1, e_2]$, then $f_{r+1} = [f_1, f_2]$). With the opposite sign convention for f_i, one has $\tau(e_i) = -f_i$ and the less uniform rule $\tau(e_\alpha) = (-1)^{ht(\alpha)} f_\alpha$. A basis of the maximal compact subalgebra is given by $k_\alpha = e_\alpha + f_\alpha$. It is convenient to define $\mathcal{G}^T = -\tau(\mathcal{G})$ for any Lie algebra element $\mathcal{G}$.

We shall also be interested in infinite-dimensional (Lorentzian) Kac-Moody algebras, for which we adopt the same conventions. In that case, the root space associated with imaginary roots might be degenerate and we therefore add an index s to account for the degeneracy, $e_\alpha \to e_{\alpha,s}$, $f_\alpha \to f_{\alpha,s}$, etc.

We shall exclusively deal in this paper with the split real forms of the Lie algebras, defined as above in terms of the same Chevalley-Serre presentation but with coefficients that are restricted to be real numbers. Remarks on the non-split case are given in the conclusions.

2.2 Invariant bilinear form

The invariant bilinear form on the Lie algebra is given for the Chevalley-Serre generators by

$$K(h_i, h_j) = \frac{2\, A_{ji}}{(\alpha_i|\alpha_i)} = \frac{2\, A_{ij}}{(\alpha_j|\alpha_j)}, \quad K(h_i, e_j) = K(h_i, f_j) = 0, \quad K(e_i, f_j) = -\frac{2\delta_{ij}}{(\alpha_i|\alpha_i)} \qquad (2.1)$$

and is extended to the full algebra by using the invariance relation $K(x, [y, z]) = K([x, y], z)$. Here, the α_i's are the simple roots. The induced bilinear form in root space is denoted by $(\cdot|\cdot)$ and given by $(\alpha_i|\alpha_j) = \frac{2A_{ij}}{(\alpha_i|\alpha_i)}$. The numbers $(\alpha_i|\alpha_i)$ are such that the product $A_{ij}\,(\alpha_i|\alpha_i)$ is

symmetric and they are normalized so that the longest roots have squared length equal to 2. One gets

$$K(h_i, e_\alpha) = K(h_i, f_\alpha) = 0 = K(e_\alpha, e_\beta) = K(f_\alpha, f_\beta), \quad K(e_\alpha, f_\beta) = -N_\alpha \delta_{\alpha\beta}, \tag{2.2}$$

where the coefficient N_α in front of $\delta_{\alpha\beta}$ in the last relation depends on the Cartan matrix (and on the precise normalization of the root vectors corresponding to higher roots – e.g., $N_\alpha = \frac{2}{(\alpha|\alpha)}$ in the Cartan-Weyl-Chevalley basis). In a representation T, the bilinear form $K(x, y)$ is proportional to the trace $\mathrm{Tr}(T(x)T(y))$.

In the finite-dimensional case, the Cartan subalgebra $\mathcal{H}$ is an Euclidean vector space. As in [24], we shall sometimes find it convenient to use then an orthogonal basis $\{H_i\}$ of the Cartan subalgebra such that

$$K(H_i, H_j) = 2\delta_{ij}. \tag{2.3}$$

It follows that $K(\vec{a}, \vec{b}) = 2\vec{a}.\vec{b}$ for $\vec{a}$, $\vec{b}$ in $\mathcal{H}$ ($\vec{a} = \sum_i a^i H_i$, $\vec{b} = \sum_i b^i H_i$) and $(\vec{\alpha}|\vec{\beta}) = \frac{1}{2}\vec{\alpha}.\vec{\beta}$ for $\vec{\alpha}$, $\vec{\beta}$ in the dual space. Here, $\vec{a}.\vec{b} = \sum_i a^i b^i$ and $\vec{\alpha}.\vec{\beta} = \sum_i \alpha_i \beta_i$ (and α_i, β_i components of $\vec{\alpha}$, $\vec{\beta}$ in the dual basis).

2.3 $K(G)$-connection

We parametrize the coset space $G/K(G)$ (with the gauge subgroup $K(G)$ acting from the left) by taking the group elements $\mathcal{V}$ in the upper-triangular "Borel gauge". The differential

$$\mathcal{G} = \mathrm{d}\mathcal{V}\mathcal{V}^{-1} \tag{2.4}$$

is in the Borel algebra, i.e., is a linear combination of the h_i's and the e_α's. It is invariant under right multiplication. One defines $\mathcal{P}$ as its symmetric part and $\mathcal{Q}$ as its antisymmetric part,

$$\mathcal{P} = \frac{1}{2}(\mathcal{G} + \mathcal{G}^T), \qquad \mathcal{Q} = \frac{1}{2}(\mathcal{G} - \mathcal{G}^T)\,. \tag{2.5}$$

$\mathcal{Q}$ is in the compact subalgebra. Under a gauge transformation $\mathcal{P}$ is covariant whereas the antisymmetric part $\mathcal{Q}$ transforms as a gauge connection,

$$\mathcal{V} \longrightarrow H\mathcal{V}\,, \quad \mathcal{P} \longrightarrow H\mathcal{P}H^{-1}\,, \quad \mathcal{Q} \longrightarrow \mathrm{d}HH^{-1} + H\mathcal{Q}H^{-1} \tag{2.6}$$

(with $H \in K(G)$).

One may parametrize $\mathcal{V}$ as $\mathcal{V} = \mathcal{V}_1\,\mathcal{V}_2$ where $\mathcal{V}_1$ is in the Cartan torus $\mathcal{V}_1 = e^{\frac{1}{2}\phi^i H_i}$ and $\mathcal{V}_2$ is in the nilpotent subgroup generated by the e_α's. In terms of this parametrization, one finds

$$\mathcal{G} = \frac{1}{2}\mathrm{d}\phi^i H_i + \sum_{\alpha\in\Delta_+} e^{\frac{1}{2}\vec{\alpha}.\vec{\phi}}\mathcal{F}_\alpha e_\alpha \tag{2.7}$$

where Δ_+ is the set of positive roots. Here, the one-forms $\mathcal{F}_\alpha$ are defined through

$$\mathrm{d}\mathcal{V}_2\mathcal{V}_2^{-1} = \sum_{\alpha\in\Delta_+} \mathcal{F}_\alpha e_\alpha. \tag{2.8}$$

In the infinite-dimensional case, there is also a sum over the multiplicity index. Thus we get

$$\mathcal{P} = \frac{1}{2}\mathrm{d}\phi^i H_i + \frac{1}{2}\sum_{\alpha\in\Delta_+} e^{\frac{1}{2}\vec{\alpha}.\vec{\phi}}\mathcal{F}_\alpha\,(e_\alpha - f_\alpha) \tag{2.9}$$

and

$$\mathcal{Q} = \frac{1}{2}\sum_{\alpha\in\Delta_+} e^{\frac{1}{2}\vec{\alpha}.\vec{\phi}}\mathcal{F}_\alpha\,k_\alpha \equiv \sum_{\alpha\in\Delta_+} \mathcal{Q}_{(\alpha)}\,k_\alpha \tag{2.10}$$

with

$$\mathcal{Q}_{(\alpha)} = \frac{1}{2} e^{\frac{1}{2}\vec{\alpha}.\vec{\phi}} \mathcal{F}_\alpha. \tag{2.11}$$

We see therefore that the one-forms $(1/2)e^{\frac{1}{2}\vec{\alpha}.\vec{\phi}}\mathcal{F}_\alpha$ appear as the components of the connection one-form of the compact subgroup $K(G)$ in the basis of the k_α's.

2.4 Lagrangian for coset models

The Lagrangian for the coset model $G/K(G)$ reads

$$\mathcal{L}_{G/K(G)} = -K\left(\mathcal{P}, \wedge * \mathcal{P}\right) \tag{2.12}$$

If we expand the Lagrangian according to (2.9), we get

$$\mathcal{L}_{G/K(G)} = -\tfrac{1}{2} {*d\vec{\phi}} \wedge d\vec{\phi} - \frac{1}{2} \sum_{\alpha \in \Delta_+} N_\alpha \, e^{\vec{\alpha}.\vec{\phi}} * \mathcal{F}_\alpha \wedge \mathcal{F}_\alpha \tag{2.13}$$

where the factor N_α is defined in (2.2).

The coupling of a field ψ transforming in a representation J of the "unbroken" subgroup $K(G)$ is straightforward. One replaces ordinary derivatives ∂_μ by covariant derivatives D_μ where

$$D_\mu \psi = \partial_\mu \psi - \sum_{\alpha \in \Delta_+} \mathcal{Q}_{\mu(\alpha)} J_\alpha \psi \tag{2.14}$$

with $\mathcal{Q}_{(\alpha)} = \mathcal{Q}_{\mu(\alpha)} dx^\mu$. In (2.14), J_α are the generators of the representation J of $K(G)$ in which ψ transforms, $J_\alpha = J(k_\alpha)$ (the generators J_α obeys the same commutation relations as k_α). This guarantees $K(G)$ – and hence G – invariance. The three-dimensional Dirac Lagrangian is thus (in flat space)

$$\bar{\psi}\gamma^\mu \left(\partial_\mu - \sum_{\alpha \in \Delta_+} \mathcal{Q}_{\mu(\alpha)} J_\alpha \right) \psi \tag{2.15}$$

2.5 Dimensional reduction of metric and exterior forms

For the purpose of being self-contained, we recall the general formulas for dimensional reduction on a torus of the metric and a $(p-1)$-form potential. We adhere to the conventions and notations of [4], which we follow without change. We consider reduction down to three spacetime dimensions.

The metric is reduced as

$$ds_D^2 = e^{\vec{s}.\vec{\phi}} ds_3^2 + \sum_{i=1}^{n} e^{2\vec{\gamma}_i.\vec{\phi}} (h^i)^2 \tag{2.16}$$

where the one-forms h^i are given by

$$h^i = dz^i + \mathcal{A}^i_{(0)j}\, dz^j + \mathcal{A}^i_{(1)} = \tilde{\gamma}^i{}_j (dz^j + \hat{\mathcal{A}}^j_{(1)}) \tag{2.17}$$

with $\tilde{\gamma}^i{}_j \equiv (\gamma^{-1})^i{}_j \equiv \delta^i_j + \mathcal{A}^i_{(0)j}$ and $\mathcal{A}^i_{(0)j}$ non vanishing only for $i < j$. The vector $\vec{\phi}$ collects the dilatons. The z^i's are internal coordinates on the torus, while $\mathcal{A}^i_{(0)j}$ and $\mathcal{A}^i_{(1)}$ are respectively scalars and 1-forms in three dimensions. Furthermore, $\vec{s} = (s_1, \ldots, s_n)$ and $\vec{\gamma}_i = \frac{1}{2}(s_1, \ldots, s_{i-1}, (2+i-D)s_i, 0, \ldots, 0)$ where

$$s_i = \sqrt{\frac{2}{(D-1-i)(D-2-i)}}$$

.

A $(p-1)$-form potential A_{p-1} decomposes as a sum of 1-forms and of scalars in three dimensions,

$$A_{p-1} = A_{(1)i_1\dots i_{p-2}} dz^{i_1} \wedge \cdots \wedge dz^{i_{p-2}} + A_{(0)i_1\dots i_{p-1}} dz^{i_1} \wedge \cdots \wedge dz^{i_{p-1}} \qquad (2.18)$$

(the 2-form component carries no degree of freedom and is dropped).

3 Dimensional reduction of the Einstein-Dirac System

3.1 Reduction of gravity

We start with the simplest case, namely, that of the coupled Einstein-Dirac system without extra fields. The bosonic sector reduces to pure gravity. To show that the Dirac field is compatible with the hidden symmetry is rather direct in this case.

Upon dimensional reduction down to $d = 3$, gravity in $D = 3 + n$ gets a symmetry group $SL(n+1)$, beyond the $SL(n)$ symmetry of the reduced dimensions. Moreover, in three dimensions the scalars describe a $SL(n+1)/SO(n+1)$ σ-model. Following (2.16), we parametrize the $D = 3 + n$ vielbein in a triangular gauge as

$$e = \begin{pmatrix} e^{\frac{1}{2}\vec{s}.\vec{\phi}} \hat{e}_\mu{}^m & e^{\vec{\gamma}_i.\vec{\phi}} \mathcal{A}_{(1)\mu}{}^i \\ 0 & M_i{}^j \end{pmatrix} \qquad (3.1)$$

with $\mu, m = 0, D-2, D-1$ and $i, j = 1..n$. We choose the non-compactified dimensions to be 0, $D-2$ and $D-1$ so that indices remain simple in formulas . The triad in three spacetime dimensions is $\hat{e}_\mu{}^m$. We denote by M the upper-triangular matrix

$$M_i{}^j = e^{\vec{\gamma}_i.\vec{\phi}} (\delta_i^j + \mathcal{A}_{(0)}{}^i{}_j) = e^{\vec{\gamma}_i.\vec{\phi}} (\gamma^{-1})^i{}_j \qquad (3.2)$$

One can check that $\det(M) = e^{-\frac{1}{2}\vec{s}.\vec{\phi}}$.

After dualizing the Kaluza-Klein vectors $\mathcal{A}_{(1)}{}^i$ into scalars χ_j as

$$e^{\vec{b}_i.\vec{\phi}} * \left(\tilde{\gamma}^i{}_j d(\gamma^j{}_m \mathcal{A}^m_{(1)}) \right) = \gamma^j{}_i\, (\mathrm{d}\chi_j) \qquad (3.3)$$

one can form the upper-triangular $(n+1) \times (n+1)$ matrix

$$\mathcal{V}^{-1} = \begin{pmatrix} M_i{}^j & \chi_i e^{\frac{1}{2}\vec{s}.\vec{\phi}} \\ 0 & e^{\frac{1}{2}\vec{s}.\vec{\phi}} \end{pmatrix} \qquad (3.4)$$

which parametrizes a $SL(n+1)/SO(n+1)$ symmetric space. With this parametrization, the three-dimensional reduced Lagrangian becomes

$$\mathcal{L}_E^{(3)} = \hat{R} * 1 - \frac{1}{2} \mathrm{Tr}\, (\mathcal{P} \wedge *\mathcal{P}) \qquad (3.5)$$

where one finds explicitly from (3.4)

$$\mathcal{G} = \mathrm{d}\mathcal{V}\mathcal{V}^{-1} = -\mathcal{V}(\mathrm{d}\mathcal{V}^{-1}) = - \begin{pmatrix} M^{-1}\mathrm{d}M & M^{-1}\mathrm{d}\chi e^{\frac{1}{2}\vec{s}.\vec{\phi}} \\ 0 & \frac{1}{2}\vec{s}.\mathrm{d}\vec{\phi} \end{pmatrix} \qquad (3.6)$$

and

$$\mathcal{P} \;=\; - \begin{pmatrix} \frac{1}{2}\left(M^{-1}\mathrm{d}M + (M^{-1}\mathrm{d}M)^T\right) & \frac{1}{2} M^{-1}\mathrm{d}\chi e^{\frac{1}{2}\vec{s}.\vec{\phi}} \\ \frac{1}{2}\left(M^{-1}\mathrm{d}\chi e^{\frac{1}{2}\vec{s}.\vec{\phi}}\right)^T & \frac{1}{2}\vec{s}.\mathrm{d}\vec{\phi} \end{pmatrix} \qquad (3.7)$$

$$\mathcal{Q} \;=\; - \begin{pmatrix} \frac{1}{2}\left(M^{-1}\mathrm{d}M - (M^{-1}\mathrm{d}M)^T\right) & \frac{1}{2} M^{-1}\mathrm{d}\chi e^{\frac{1}{2}\vec{s}.\vec{\phi}} \\ -\frac{1}{2}\left(M^{-1}\mathrm{d}\chi e^{\frac{1}{2}\vec{s}.\vec{\phi}}\right)^T & 0 \end{pmatrix} . \qquad (3.8)$$

3.2 Adding spinors

We now couple a Dirac spinor to gravity in $D = 3 + n$ dimensions:

$$\mathcal{L} = \mathcal{L}_E + \mathcal{L}_D \tag{3.9}$$

with $\mathcal{L}_E$ the Einstein Lagrangian and $\mathcal{L}_D$ the Dirac Lagrangian,

$$\mathcal{L}_D = e\overline{\psi}\not{D}\psi = e\overline{\psi}\gamma^\Sigma\left(\partial_\Sigma - \frac{1}{4}\omega_{\Sigma,AB}\gamma^{AB}\right)\psi \tag{3.10}$$

e is the determinant of the vielbein and $\Sigma, A = 0, ..., D_{max} - 1$. The indices A, B, ... are internal indices and Σ, Ω are spacetime indices, while ω is the spin connection, which can be computed from the vielbein:

$$\begin{aligned}\omega_{\Sigma,AB} &= \frac{1}{2}e_A{}^\Omega(\partial_\Omega e_{\Sigma B} - \partial_\Sigma e_{\Omega B}) - \frac{1}{2}e_B{}^\Omega(\partial_\Omega e_{\Sigma A} - \partial_\Sigma e_{\Omega A}) \\ &\quad -\frac{1}{2}e_A{}^\Pi e_B{}^\Omega(\partial_\Omega e_{\Pi C} - \partial_\Pi e_{\Omega C})e_\Sigma{}^C\ .\end{aligned} \tag{3.11}$$

We perform a dimensional reduction, with the vielbein parametrized by (3.1), by imposing the vanishing of derivatives ∂_Σ for $\Sigma \geq 3$. We also rescale ψ by a power f of the determinant of the reduced part of the vielbein M:

$$\hat{\psi} = e^{-\frac{1}{2}f\vec{s}.\vec{\phi}}\psi\ . \tag{3.12}$$

After reassembling the various terms, we find that the reduced Dirac Lagrangian can be written as

$$\begin{aligned}\mathcal{L}_D^{(3)} &= e^{(\frac{1}{2}+f)\vec{s}.\vec{\phi}}\hat{e}\overline{\hat{\psi}}\hat{\not{D}}\hat{\psi} + \left(\frac{1}{2}f + \frac{1}{4}\right)e^{(\frac{1}{2}+f)\vec{s}.\vec{\phi}}\hat{e}\,\partial_\mu(\vec{s}.\vec{\phi})\,\overline{\hat{\psi}}\gamma^\mu\hat{\psi} \\ &\quad + \frac{1}{8}e^{(\frac{1}{2}+f)\vec{s}.\vec{\phi}}\hat{e}\left(M^{-1}{}_j{}^k\partial_\mu M_k{}^i - M^{-1}{}_i{}^k\partial_\mu M_k{}^j\right)\overline{\hat{\psi}}\gamma^\mu\gamma^{ij}\hat{\psi} \\ &\quad + \frac{1}{8}e^{f\vec{s}.\vec{\phi}}\hat{e}\,\hat{e}_m{}^\mu e_n{}^\Omega\Big(\partial_\mu(\mathcal{A}_{(1)\Omega}{}^j M^{-1}{}_j{}^k) - \partial_\Omega(\mathcal{A}_{(1)\mu}{}^j M^{-1}{}_j{}^k)\Big)M_k{}^i\,\overline{\hat{\psi}}\gamma^i\gamma^{mn}\hat{\psi}\end{aligned} \tag{3.13}$$

where $\hat{e}$ is the determinant of the dreibein. Note that the numerical matrices γ^M (with M an internal index) are left unchanged in the reduction process, but this is not the case for γ^Σ (with Σ a spacetime index). Indeed, γ^Σ for $\Sigma = \mu$ has to be understood as $e_M{}^\mu\gamma^M$ in D dimensions, and as $\hat{e}_m{}^\mu\gamma^m$ in 3 dimensions. Nevertheless, we do not put hats on three-dimensional γ-matrices with a spatial index as no confusion should arise.

Dualizing $\mathcal{A}_{(1)}$ according to (3.3), we get

$$\begin{aligned}\mathcal{L}_D^{(3)} &= e^{(\frac{1}{2}+f)\vec{s}.\vec{\phi}}\hat{e}\overline{\hat{\psi}}\hat{\not{D}}\hat{\psi} + \left(\frac{1}{2}f + \frac{1}{4}\right)e^{(\frac{1}{2}+f)\vec{s}.\vec{\phi}}\hat{e}\,\partial_\mu(\vec{s}.\vec{\phi})\,\overline{\hat{\psi}}\gamma^\mu\hat{\psi} \\ &\quad + \frac{1}{8}e^{(\frac{1}{2}+f)\vec{s}.\vec{\phi}}\hat{e}\left(M^{-1}{}_j{}^k\partial_\mu M_k{}^i - M^{-1}{}_i{}^k\partial_\mu M_k{}^j\right)\overline{\hat{\psi}}\gamma^\mu\gamma^{ij}\hat{\psi} \\ &\quad + \frac{1}{8}e^{(1+f)\vec{s}.\vec{\phi}}\hat{e}\,\epsilon_{mnp}\hat{e}_p{}^\mu M^{-1}{}_i{}^j\partial_\mu\chi_j\overline{\hat{\psi}}\gamma^i\gamma^{mn}\hat{\psi}\ .\end{aligned} \tag{3.14}$$

If we choose the scaling power of the spinor as

$$f = -\frac{1}{2}\ , \tag{3.15}$$

the Lagrangian simplifies to

$$\begin{aligned}\mathcal{L}_D^{(3)} &= \hat{e}\overline{\hat{\psi}}\hat{\not{D}}\hat{\psi} + \frac{1}{8}\hat{e}\left(M^{-1}{}_j{}^k\partial_\mu M_k{}^i - M^{-1}{}_i{}^k\partial_\mu M_k{}^j\right)\overline{\hat{\psi}}\gamma^\mu\gamma^{ij}\hat{\psi} \\ &\quad + \frac{1}{4}\hat{e}\,e^{\frac{1}{2}\vec{s}.\vec{\phi}}M^{-1}{}_i{}^j\partial_\mu\chi_j\overline{\hat{\psi}}\gamma^\mu\hat{\gamma}\gamma^i\hat{\psi}\end{aligned} \tag{3.16}$$

where we have used the notation $\hat{\gamma} = \gamma^0 \gamma^{D-2} \gamma^{D-1}$.

In fact, the three-dimensional Lagrangian can be rewritten using a covariant derivative including a connection with respect to the gauge group $SO(n+1)$:

$$\mathcal{L}_D^{(3)} = \hat{e}\bar{\hat{\psi}} \nabla\!\!\!\!/\, \hat{\psi} \tag{3.17}$$

with

$$\nabla_\mu = \partial_\mu - \frac{1}{4}\hat{\omega}_{\mu,mn}\gamma^{mn} - \frac{1}{2}\mathcal{Q}_{\mu,ij}J^{ij} \tag{3.18}$$

where $\mathcal{Q}$ is the $SO(n+1)$ connection (3.8), acting on Dirac spinors through

$$J_{ij} = \frac{1}{2}\gamma^{ij}\,, \qquad J_{i(n+1)} = \frac{1}{2}\hat{\gamma}\gamma^i \qquad (i,j = 1..n)\ . \tag{3.19}$$

These matrices define a spinorial representation of $SO(n+1)$. The commutations relations are indeed

$$\begin{array}{rcl} \left[\frac{1}{2}\gamma^{ij}, \frac{1}{2}\gamma^{kl}\right] & = & 0 \\ \left[\frac{1}{2}\gamma^{ij}, \frac{1}{2}\gamma^{ik}\right] & = & -\frac{1}{2}\gamma^{jk} \\ \left[\frac{1}{2}\hat{\gamma}\gamma^{i}, \frac{1}{2}\gamma^{jk}\right] & = & 0 \\ \left[\frac{1}{2}\hat{\gamma}\gamma^{i}, \frac{1}{2}\gamma^{ij}\right] & = & \frac{1}{2}\hat{\gamma}\gamma^{j} \\ \left[\frac{1}{2}\hat{\gamma}\gamma^{i}, \frac{1}{2}\hat{\gamma}\gamma^{j}\right] & = & -\frac{1}{2}\gamma^{ij} \end{array} \tag{3.20}$$

where different indices are supposed to be distinct. Equivalently, we can remark that $\hat{\gamma}$ commutes with γ^m for $m = 0, D-2, D-1$ and anticommutes with γ^i for $i = 1, ..., n$. As we have also $\hat{\gamma}^2 = 1$, it follows that γ^a's for $a = 1, ..., n$ and $\hat{\gamma}$ generate an internal $Spin(n+1)$ Clifford algebra, commuting with the spacetime Clifford algebra generated by $\gamma^m, m = 0, D-2, D-1$.

In other words, the $Spin(n+2,1)$ representation of Dirac fermions in dimension $D = 3+n$ is reduced to a $Spin(2,1) \times Spin(n+1)$ representation in dimension 3, ensuring that the Dirac fermions are compatible with the hidden symmetry. Note that if D is even, one can impose chirality conditions on the spin 1/2 field in D dimensions. One gets in this way a chiral spinor of $SO(n+1)$ after dimensional reduction.

3.3 Explicit Borel decomposition

One may write the Lagrangian in the form (2.13) by making a full Borel parametrization of the matrix M. The algebra element $\mathcal{G}$ reads

$$\mathcal{G} = \frac{1}{2}\mathrm{d}\vec{\phi}.\vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}.\vec{\phi}}\mathcal{F}_{(1)}{}^i{}_j e_{b_{ij}} + \sum_i e^{-\frac{1}{2}\vec{b}_i.\vec{\phi}}\mathcal{G}_{(1)}{}^i e_{b_i} \tag{3.21}$$

where the dilaton vectors are given by

$$\vec{b}_i = -\vec{s} + 2\vec{\gamma}_i\ , \qquad \vec{b}_{ij} = 2\vec{\gamma}_i - 2\vec{\gamma}_j \tag{3.22}$$

and where the “field strengths" $\mathcal{F}^i_{(1)j}$ and $\mathcal{G}_{(1)}{}^i$ (which are also the $SO(n+1)$ connections) are

$$\mathcal{F}^i_{(1)j} = \gamma^k{}_j\, d\mathcal{A}^i_{(0)k}, \qquad \mathcal{G}_{(1)}{}^i = \gamma^j{}_i \mathrm{d}\chi_j \tag{3.23}$$

One has

$$e^{\vec{b}_i.\vec{\phi}} * \mathcal{F}_{(2)}{}^i = \mathcal{G}_{(1)}{}^i \tag{3.24}$$

with

$$\mathcal{F}^i_{(2)} = \tilde{\gamma}^i{}_j d(\gamma^j{}_m \mathcal{A}^m_{(1)}) \tag{3.25}$$

The positive roots of $SL(n+1)$ are $(\vec{b}_{ij}, -\vec{b}_i)$ and the corresponding root vectors $e_{b_{ij}}$ and e_{b_i} are the multiple commutators of the generators e_i not involving e_1 (for $e_{b_{ij}}$) or involving e_1 (for e_{b_i}), i.e., $e_{b_{ii+1}} = e_{i+1}$ $(i = 1, \cdots, n-1)$, $e_{b_{ij}} = [e_i, [e_{i+1}, [...[e_{j-2}, e_j] \cdots]$ $(i, j = 2, \cdots, n,\ i+1 < j)$, $e_{b_i} = [e_1, e_{b_{2i}}]$ $(i \geq 3)$, $e_{b_n} = e_n$. These root vectors are such that the normalization factors N_α are all equal to one.

The Lagrangian reads

$$\begin{aligned}
\mathcal{L}^{(3)}_{D_n} &= R *\mathbb{1} - \tfrac{1}{2} {*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2} \sum_i e^{-\vec{b}_i \cdot \vec{\phi}} *\mathcal{G}_{(1)i} \wedge \mathcal{G}_{(1)i} - \tfrac{1}{2} \sum_{i<j} e^{\vec{b}_{ij} \cdot \vec{\phi}} *\mathcal{F}^i_{(1)j} \wedge \mathcal{F}^i_{(1)j} \\
&\quad + \hat{e} \bar{\hat{\psi}} \gamma^\mu \left(\partial_\mu - \frac{1}{4} \hat{\omega}_{\mu,mn} \gamma^{mn} - \tfrac{1}{4} \sum_{i<j} e^{\frac{1}{2} \vec{b}_{ij} \cdot \vec{\phi}} \mathcal{F}^i_{(1)j} \gamma^{ij} - \tfrac{1}{4} \sum_i e^{-\frac{1}{2} \vec{b}_i \cdot \vec{\phi}} \mathcal{G}_{(1)i} \hat{\gamma} \gamma^i \right) \psi
\end{aligned} \tag{3.26}$$

4 D_n case

4.1 Bosonic sector

Following [4], we consider the gravitational lagrangian $\mathcal{L}_E$ with an added three form field strength $F_{(3)}$ coupled to a dilaton field φ,

$$\mathcal{L} = R *\mathbb{1} - \tfrac{1}{2} {*d\varphi} \wedge d\varphi - \tfrac{1}{2} e^{a\varphi} *F_{(3)} \wedge F_{(3)} \tag{4.1}$$

in the dimension $D_{max} = n+2$, where the coupling constant a is given by $a^2 = 8/(D_{max} - 2)$. Upon toroidal reduction to $D = 3$, this yields the Lagrangian

$$\begin{aligned}
\mathcal{L}^{(3)} &= R *\mathbb{1} - \tfrac{1}{2} {*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2} \sum_i e^{\vec{b}_i \cdot \vec{\phi}} *\mathcal{F}^i_{(2)} \wedge \mathcal{F}^i_{(2)} - \tfrac{1}{2} \sum_{i<j} e^{\vec{b}_{ij} \cdot \vec{\phi}} *\mathcal{F}^i_{(1)j} \wedge \mathcal{F}^i_{(1)j} \\
&\quad - \tfrac{1}{2} \sum_i e^{\vec{a}_i \cdot \vec{\phi}} *F_{(2)i} \wedge F_{(2)i} - \tfrac{1}{2} \sum_{i<j} e^{\vec{a}_{ij} \cdot \vec{\phi}} *F_{(1)ij} \wedge F_{(1)ij} \ .
\end{aligned} \tag{4.2}$$

Note that here $\vec{\phi}$ denotes now the set of dilatons $(\phi_1, \phi_2, \ldots, \phi_{D_{max}-3})$, augmented by φ (the dilaton in D_{max} dimensions) as a zeroth component; $\vec{\phi} = (\varphi, \phi_1, \phi_2, \ldots, \phi_{D_{max}-3})$. The dilaton vectors entering the exponentials in the Lagrangian are given by

$$\vec{a}_i = -2\vec{\gamma}_i - \vec{s}\ , \qquad \vec{a}_{ij} = -2\vec{\gamma}_i - 2\vec{\gamma}_j\ , \qquad \vec{b}_i = -\vec{s} + 2\vec{\gamma}_i\ , \qquad \vec{b}_{ij} = 2\vec{\gamma}_i - 2\vec{\gamma}_j\ . \tag{4.3}$$

augmented by a zeroth component that is equal to the constant a in the case of $\vec{a}_i$ and $\vec{a}_{ij}$, and is equal to zero in the case of $\vec{b}_i$ and $\vec{b}_{ij}$. The field strengths are given by

$$\begin{aligned}
\mathcal{F}^i_{(2)} &= \tilde{\gamma}^i{}_j d(\gamma^j{}_m \mathcal{A}^m_{(1)})\ , \\
\mathcal{F}^i_{(1)j} &= \gamma^k{}_j\, d\mathcal{A}^i_{(0)k}\ , \\
F_{(2)i} &= \gamma^k{}_i (dA_{(1)k} + \gamma^j{}_m dA_{(0)kj} \wedge \hat{\mathcal{A}}^m_{(1)}), \\
F_{(1)ij} &= \gamma^k{}_i \gamma^m{}_j dA_{(0)km}.
\end{aligned}$$

After dualising the 1-form potentials $\mathcal{A}^i_{(1)}$ and $A_{(1)i}$ to axions χ_i and ψ^i respectively, the three-dimensional Lagrangian (4.2) can written as the purely scalar Lagrangian

$$\begin{aligned}
\mathcal{L}^{(3)}_{D_n} &= R *\mathbb{1} - \tfrac{1}{2} {*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2} \sum_i e^{-\vec{b}_i \cdot \vec{\phi}} *\mathcal{G}_{(1)i} \wedge \mathcal{G}_{(1)i} - \tfrac{1}{2} \sum_{i<j} e^{\vec{b}_{ij} \cdot \vec{\phi}} *\mathcal{F}^i_{(1)j} \wedge \mathcal{F}^i_{(1)j} \\
&\quad - \tfrac{1}{2} \sum_i e^{-\vec{a}_i \cdot \vec{\phi}} *G^i_{(1)} \wedge G^i_{(1)} - \tfrac{1}{2} \sum_{i<j} e^{\vec{a}_{ij} \cdot \vec{\phi}} *F_{(1)ij} \wedge F_{(1)ij}\ ,
\end{aligned} \tag{4.4}$$

where the dualised field strengths are given by

$$\begin{aligned} e^{\vec{b}_i\cdot\vec{\phi}} *\mathcal{F}_{(2)}{}^i \equiv \mathcal{G}_{(1)}{}^i &= \gamma^j{}_i(d\chi_j - A_{(0)kj}\, d\psi^k) \ , \\ G^i_{(1)} &= e^{-\vec{\gamma}_i\cdot\vec{\phi}} M^i{}_j\, d\psi^j \ . \end{aligned} \tag{4.5}$$

Note the modification of $\mathcal{G}_{(1)}{}^i$ due to the coupling of $\mathcal{A}^i_{(1)}$ to the 2-form variables. The positive roots of D_n are given by $(\vec{b}_{ij}, -\vec{b}_i, \vec{a}_{ij}, -\vec{a}_i)$, the simple roots being $\vec{a}_{12}$, $\vec{b}_{i,i+1}$ $(i \leq n-1)$ and $-\vec{a}_n$ [4]. The three-dimensional Lagrangian (4.4) describes a $SO(n,n)/(SO(n)\times SO(n))$ σ-model in the Borel gauge coupled to gravity. The field strength of this σ-model is

$$\begin{aligned} \mathcal{G} = \frac{1}{2}\mathrm{d}\vec{\phi}.\vec{H} &+ \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}} \mathcal{F}_{(1)}{}^i{}_j e_{b_{ij}} + \sum_i e^{-\frac{1}{2}\vec{b}_i\cdot\vec{\phi}} \mathcal{G}_{(1)}{}^i e_{b_i} \\ &+ \sum_{i<j} e^{\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}} F_{(1)ij} e_{a_{ij}} + \sum_i e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\phi}} G_{(1)}{}^i e_{a_i} \ . \end{aligned} \tag{4.6}$$

$\vec{h}$ is the vector of Cartan generators and the notations e_{a_i}, $e_{a_{ij}}, e_{b_i}$ and $e_{b_{ij}}$ are explained in appendix A.

4.2 Fermions

We want to add Dirac fermion in D_{max}, with a coupling which reduces to $SO(2,1)\times(SO(n)\times SO(n))$. The coupling to gravitational degrees of freedom is already fixed to the spin connection by invariance under reparametrization; we know from the first section that it reduces to the $SO(2,1)\times SO(n)$ connection in $D=3$. From the structure of the theory, we know that the fermions must have linear couplings with the 3-form F_3. Indeed, the $D=3$ couplings must be of the following form

$$\hat{e}\bar{\hat{\psi}}\gamma^\mu \left(\partial_\mu - \frac{1}{4}\hat{\omega}_{\mu,mn}\gamma^{mn} - \mathcal{Q}_{\mu(\alpha)}J^{(\alpha)}\right)\psi \tag{4.7}$$

where $\mathcal{Q}$ can be read off from (4.6) above and the $J^{(\alpha)}$'s are a representation of $SO(n)\times SO(n)$. The possible Lorentz-covariant coupling of this kind are the Pauli coupling and its dual,

$$-\sqrt{-g}e^{\frac{1}{2}a\varphi}\bar{\psi}\frac{1}{3!}(\alpha\gamma^{ABC} + \beta\gamma^{ABC}\gamma)F_{(3)\,ABC}\psi \tag{4.8}$$

where α and β are arbitrary constants, which will be determined below. The dilaton dependence is fixed so as to reproduce the roots $\vec{a}_i$ and $\vec{a}_{ij}$ in the exponentials in front of the fermions in the expressions below. The matrix γ is the product of all gamma matrices $\gamma=\gamma^0\gamma^1...\gamma^{D-1}$. One has $\gamma^2 = -(-1)^{[\frac{D}{2}]}$. Notice that in odd dimensions this matrix is proportional to the identity and therefore we can put $\beta=0$ without loss of generality. Thus we add to the bosonic lagrangian (4.1) the following term,

$$\mathcal{L}_\psi = \sqrt{-g}\bar{\psi}(\gamma^\mu\partial_\mu - \frac{1}{4}\omega_{\mu,mn}\gamma^{mn} - \frac{1}{3!}(\alpha\gamma^{ABC} + \beta\gamma^{ABC}\gamma)e^{\frac{1}{2}a\varphi}F_{(3)\,ABC})\psi \tag{4.9}$$

Upon toroidal reduction to $D=3$, the last term of (4.9) becomes,

$$-\frac{1}{3!}\sqrt{-\hat{g}}\bar{\hat{\psi}}\Big(e^{\frac{1}{2}\vec{a}_i\cdot\vec{\phi}}3(\alpha\gamma^{ab}\gamma^i + \beta\gamma^{abi}\gamma)F_{(2)i\ ab} + e^{\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}3!(\alpha\gamma^a\gamma^j\gamma^i + \beta\gamma^a\gamma^j\gamma^i\gamma)F_{(1)ji\ a}\Big)\hat{\psi} \tag{4.10}$$

Let us dualize the 2 form field strengths. By using the relation $\epsilon_{cab}\gamma^{ab} = 2\gamma_c\hat{\gamma}$, we get for the dimensional reduction of the whole lagrangian (4.9) (with dualisation of the $\mathcal{F}^{(i)}$'s and using the

results of the pure gravitational case),

$$\begin{aligned}
\mathcal{L}_\psi^{(3)} = \sqrt{-\hat{g}}\bar{\hat{\psi}}\gamma_c(\gamma^\mu\partial_\mu &- \frac{1}{4}\hat{\omega}_{\mu,mn}\gamma^{mn} \\
&- \frac{1}{2}e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\phi}}\Gamma_{\vec{a}_i}G_i^c - \frac{1}{2}e^{\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}\Gamma_{\vec{a}_{ij}}F^c_{(1)ij} \\
&- \frac{1}{2}e^{-\frac{1}{2}\vec{b}_i\cdot\vec{\phi}}\Gamma_{b_i}\mathcal{G}_i^c - \frac{1}{2}e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\Gamma_{b_{ij}}\mathcal{F}^c_{(1)ij})\hat{\psi}
\end{aligned} \tag{4.11}$$

where

$$\begin{aligned}
&\Gamma_{\vec{a}_i} = 2(\alpha\hat{\gamma}\gamma^i + \beta\hat{\gamma}\gamma^i\gamma), \qquad \Gamma_{\vec{a}_{ij}} = 2(\alpha\gamma^i\gamma^j + \beta\gamma^i\gamma^j\gamma), \\
&\Gamma_{\vec{b}_i} = \tfrac{1}{2}\hat{\gamma}\gamma^i, \qquad\quad \Gamma_{\vec{b}_{ij}} = \tfrac{1}{2}\gamma^i\gamma^j
\end{aligned} \tag{4.12}$$

We have to compare this expression with

$$\sqrt{-\hat{g}}\bar{\hat{\psi}}\gamma_c(\gamma^\mu\partial_\mu - \frac{1}{4}\hat{\omega}_{\mu,mn}\gamma^{mn} - \mathcal{Q}^{c(\alpha)}J^{(\alpha)})\psi \tag{4.13}$$

where $\mathcal{Q}_{\mu(\alpha)}$ are the coefficients of the $K(SO(n,n)) = SO(n)\times SO(n)$ gauge field. From (4.6), we find that

$$\begin{aligned}
\mathcal{Q} =&\frac{1}{2}\sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\mathcal{F}_{(1)}{}^i{}_j(e_{b_{ij}} + f_{b_{ij}}) + \frac{1}{2}\sum_i e^{-\frac{1}{2}\vec{b}_i\cdot\vec{\phi}}\mathcal{G}_{(1)}{}^i(e_{b_i} + f_{b_i}) \\
&+ \frac{1}{2}\sum_{i<j} e^{\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}F_{(1)ijk}(e_{a_{ij}} + f_{a_{ij}}) + \frac{1}{2}\sum_i e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\phi}}G_{(1)}{}^i(e_{a_i} + f_{a_i}) \ .
\end{aligned} \tag{4.14}$$

The commutation relations of the $k_{(\alpha)}$'s are explicitly given in appendix A. One has to fix the values of α and β such that the generators $\Gamma_{\vec{a}_i}$, $\Gamma_{\vec{b}_i}$, $\Gamma_{\vec{a}_{ij}}$ and $\Gamma_{\vec{b}_{ij}}$ obey these same commutation relations. The conditions we found are

$$\alpha^2 + \beta^2\gamma^2 = \frac{1}{16}\,, \qquad \alpha\beta = 0. \tag{4.15}$$

In odd dimension, we have set $\beta = 0$. This implies $\alpha = \pm\frac{1}{4}$. We get for each choice of α a representation which is trivial for either the left or the right $SO(n)$ factor of the compact gauge group. With $\beta = 0$, (4.12) generates indeed $SO(n)$, as our analysis of the gravitational sector has already indicated.

In even dimension, the choices $\beta = 0, \alpha = \pm\frac{1}{4}$ are still solutions to (4.15), but in addition one can have $\alpha = 0$, $\beta = \pm\frac{\iota}{4}$, where the constant ι is 1 or i such that $(\iota\gamma)^2 = 1$. In this case, (4.12) combines with the gravitational $SO(n)$ to give a $SO(n)\times SO(n)$ representation which is nontrivial on both factors. The two factors $SO(n)_\pm$ are generated in the spinorial space by the matrices

$$\begin{aligned}
&\tfrac{1}{4}(1 \pm \iota\gamma)\gamma^{ij} \\
&\tfrac{1}{4}(1 \pm \iota\gamma)\hat{\gamma}\gamma^i
\end{aligned} \tag{4.16}$$

The (reduced) gravitational sector is given by the diagonal $SO(n)$. If one imposes a chirality condition in D_{max} dimensions, the solution with $\beta = 0$ and the solution with $\alpha = 0$ are of course equivalent and the representation is trivial on one of the $SO(n)$.

This completes the proof that the Dirac spinors are compatible with the D_n hidden symmetry.

5 E_n sequence

5.1 E_8 – bosonic

We consider now the bosonic part of 11-dimensional supergravity, *i.e.* gravity coupled to a 3-form in 11 dimensions with the specific value of the Chern-Simons term dictated by supersymmetry. We denote the 3-form $A_{(3)}$ and its field strength $F_{(4)} = \mathrm{d}A_{(3)}$.

The Lagrangian is [14]

$$\mathcal{L} = R*\mathbb{1} - \frac{1}{2}*F_{(4)} \wedge F_{(4)} - \frac{1}{3!}F_{(4)} \wedge F_{(4)} \wedge A_{(3)} \ . \tag{5.1}$$

Prior to dualization, the 3-form term of the Lagrangian reduces in three dimensions to

$$\begin{aligned} \hat{\mathcal{L}}_3 &= -\frac{1}{2}\sum_{i<j<k} e^{\vec{a}_{ijk}.\vec{\phi}}(F_{(1)ijk})^2 - \frac{1}{4}\hat{e}\sum_{i<j} e^{\vec{a}_{ij}.\vec{\phi}}(F_{(2)ij})^2 \\ &\qquad -\frac{1}{144}\mathrm{d}A_{(0)ijk} \wedge \mathrm{d}A_{(0)lmn} \wedge A_{(1)pq}\epsilon^{ijklmnpq} \end{aligned} \tag{5.2}$$

In addition to the gravitational degrees of freedom described in section 3, we have 56 scalars $A_{(0)ijk}$ and 28 1-forms $A_{(1)ij} = A_{\mu(i+2)(j+2)}\mathrm{d}x^\mu$, with $i,j,k = 1..8$. The reduced field strength are defined as

$$F_{(1)ijk} = \gamma^l{}_i\gamma^m{}_j\gamma^n{}_k\,\mathrm{d}A_{(0)lmn} \tag{5.3}$$

$$F_{(2)ij} = \gamma^k{}_i\gamma^l{}_j\left(\mathrm{d}A_{(1)kl} - \gamma^m{}_n\,\mathrm{d}A_{(0)klm} \wedge \mathcal{A}_{(1)}{}^n\right) \ . \tag{5.4}$$

The 1-forms $A_{(1)ij}$ are then dualized into scalars λ^{kl}:

$$e^{\vec{a}_{ij}.\vec{\phi}}*F_{(2)ij} = G_{(1)}{}^{ij} = (\gamma^{-1})^i{}_k(\gamma^{-1})^j{}_l\left(\mathrm{d}\lambda^{kl} + \frac{1}{72}\mathrm{d}A_{(0)mnp}A_{(0)qrs}\epsilon^{klmnpqrs}\right) \ . \tag{5.5}$$

Moreover, the gravitational duality relation (3.3) has to be modified to take into account the 3-form degrees of freedom

$$e^{\vec{b}_i.\vec{\phi}}*\mathcal{F}_{(2)}{}^i = \mathcal{G}_{(1)}{}^i = \gamma^j{}_i\left(\mathrm{d}\chi_j - \frac{1}{2}A_{(0)jkl}\mathrm{d}\lambda^{kl} - \frac{1}{432}\mathrm{d}A_{(0)klm}A_{(0)npq}A_{(0)rsj}\epsilon^{klmnpqrs}\right) \ . \tag{5.6}$$

Taking all this into account, the full 3-dimensional Lagrangian can be written as

$$\begin{aligned} \hat{\mathcal{L}} = R*\mathbb{1} &- \frac{1}{2}\mathrm{d}\vec{\phi}\wedge*\mathrm{d}\vec{\phi} - \frac{1}{2}\hat{e}\sum_{i<j} e^{\vec{b}_{ij}.\vec{\phi}}\mathcal{F}_{(1)}{}^i{}_j \wedge*\mathcal{F}_{(1)}{}^i{}_j - \frac{1}{2}\hat{e}\sum_i e^{-\vec{b}_i.\vec{\phi}}\mathcal{G}_{(1)}{}^i \wedge*\mathcal{G}_{(1)}{}^i \\ &-\frac{1}{2}\hat{e}\sum_{i<j<k} e^{\vec{a}_{ijk}.\vec{\phi}}F_{(1)ijk} \wedge*F_{(1)ijk} - \frac{1}{2}\hat{e}\sum_{i<j} e^{-\vec{a}_{ij}.\vec{\phi}}G_{(1)}{}^{ij} \wedge*G_{(1)}{}^{ij} \end{aligned} \tag{5.7}$$

which describes a $E_{8(8)}/SO(16)$ σ-model coupled to gravity [2, 27, 24], in the Borel gauge, with field strength

$$\begin{aligned} \mathcal{G} = \frac{1}{2}\mathrm{d}\vec{\phi}.\vec{H} &+ \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}.\vec{\phi}}\mathcal{F}_{(1)}{}^i{}_j e_{ij} + \sum_i e^{-\frac{1}{2}\vec{b}_i.\vec{\phi}}\mathcal{G}_{(1)}{}^i e_i \\ &+ \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk}.\vec{\phi}}F_{(1)ijk}\tilde{e}_{ijk} + \sum_{i<j} e^{-\frac{1}{2}\vec{a}_{ij}.\vec{\phi}}G_{(1)}{}^{ij}\tilde{e}_{ij} \ . \end{aligned} \tag{5.8}$$

The explicit expressions for the couplings $\vec{a}_{ijk}$ and $\vec{a}_{ij}$ are (see [24])

$$\vec{a}_{ijk} = -2(\vec{\gamma}_i + \vec{\gamma}_j + \vec{\gamma}_k), \quad \vec{a}_{ij} = -2(\vec{\gamma}_i + \vec{\gamma}_j) - \vec{s}. \tag{5.9}$$

The positive roots are $\vec{b}_{ij}$, $-\vec{b}_i$, $\vec{a}_{ijk}$ and $-\vec{a}_{ij}$. The elements e_{ij} $(i < j)$, e_i, $\tilde{e}_{ijk}$ and $\tilde{e}_{ij}$ (with antisymmetry over the indices for the two last cases) are the raising operators. Note that e_{ij} and e_i generate the $SL(9)$ subalgebra coming from the gravitational sector. In addition, there are lowering operators f_{ij}, f_i, $\tilde{f}_{ijk}$ and $\tilde{f}_{ij}$. We give all the commutation relations in that basis of E_8 in appendix B.

5.2 E_8 – fermions

The maximal compact subgroup of E_8 is $SO(16)$; its generators are given in appendix B. We want to add Dirac fermions in $D = 11$, with a coupling which reduces to a $SO(2,1) \times SO(16)$-covariant derivative in three dimensions. The coupling to gravitational degrees of freedom is already fixed to the spin connection by invariance under reparametrizations; we know from the first section that it reduces to the relevant $SO(2,1) \times SO(9)$ connection in $D = 3$.

From the structure of the reduced theory, we know that the fermion must have a linear coupling to the 4-form $F_{(4)}$. The only Lorentz-covariant coupling of this kind for a single Dirac fermion in $D = 11$ is a Pauli coupling

$$ea\frac{1}{4!}\overline{\psi}F_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}\psi \tag{5.10}$$

where a is a constant. Indeed in odd dimensions, the product of all γ matrices is proportional to the identity, so the dual coupling is not different:

$$\frac{1}{7!}(*F)_{\mu_1\ldots\mu_7}\gamma^{\mu_1\ldots\mu_7} = \frac{1}{4!}F_{\mu_1\ldots\mu_4}\gamma^{\mu_1\ldots\mu_4}\ . \tag{5.11}$$

Thus we add to the bosonic Lagrangian (5.1) the fermionic term

$$\mathcal{L}_\psi = e\overline{\psi}\left(\gamma^\mu\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma^\mu\gamma^{ab} - \frac{1}{4!}aF_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}\right)\psi \tag{5.12}$$

where γ matrices with greek, curved indices must be understood as $\gamma^\mu = e_a{}^\mu\gamma^a$.

Dimensional reduction to $D = 3$ leads to

$$\mathcal{L}_\psi^{(3)} = \mathcal{L}_D^{(3)} - \hat{e}a\frac{1}{3!}e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}F_{(1)\mu ijk}\overline{\hat{\psi}}\gamma^\mu\gamma^{ijk}\hat{\psi} - \hat{e}a\frac{1}{2.2}e^{\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}F_{(2)\mu\nu ij}\overline{\hat{\psi}}\gamma^{\mu\nu}\gamma^{ij}\hat{\psi} \tag{5.13}$$

where $\mathcal{L}_D^{(3)}$ is part not containing the 3-form computed previously in (3.16), and with the same rescaling of ψ into $\hat{\psi}$. Dualisation (5.5) of $F_{(2)ij}$ can be written as

$$\frac{1}{2}e^{\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}F_{(2)\mu\nu ij}\gamma^{\mu\nu} = e^{-\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}G_{(1)\mu ij}\gamma^\mu\hat{\gamma}\ . \tag{5.14}$$

It gives the fully dualised fermionic term

$$\mathcal{L}_\psi^{(3)} = \mathcal{L}_D^{(3)} - \hat{e}a\frac{1}{3!}e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}F_{(1)\mu ijk}\overline{\hat{\psi}}\gamma^\mu\gamma^{ijk}\hat{\psi} - \hat{e}a\frac{1}{2}e^{-\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}G_{(1)\mu ij}\overline{\hat{\psi}}\gamma^\mu\hat{\gamma}\gamma^{ij}\hat{\psi}\ . \tag{5.15}$$

We have to compare this expression to

$$\hat{e}\overline{\hat{\psi}}\gamma^\mu\left(\partial_\mu - \frac{1}{4}\hat{\omega}_{\mu,mn}\gamma^{mn} - \mathcal{Q}^{\mu(\alpha)}J^{(\alpha)}\right)\psi. \tag{5.16}$$

From (5.8), we have

$$\begin{aligned}\mathcal{Q} =& \frac{1}{2}\sum_{i<j}e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\mathcal{F}_{(1)}{}^i{}_j(e_{ij}+f_{ij}) + \frac{1}{2}\sum_i e^{-\frac{1}{2}\vec{b}_i\cdot\vec{\phi}}\mathcal{G}_{(1)}{}^i(e_i+f_i)\\ &+\frac{1}{2}\sum_{i<j<k}e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}F_{(1)ijk}(\tilde{e}_{ijk}+\tilde{f}_{ijk}) + \frac{1}{2}\sum_{i<j}e^{-\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}G_{(1)}{}^{ij}(\tilde{e}_{ij}+\tilde{f}_{ij})\ .\end{aligned} \tag{5.17}$$

In fact, we have precisely the correct gauge connection that appears in (5.15). We have only to check that the products of gamma matrices that multiply the connection in (5.15) satisfy the correct commutation relations. Using the commutation relations of the compact generators $k_{ij} = e_{ij} + f_{ij}$ $(i < j)$, $k_i = e_i + f_i$, $\tilde{k}_{ijk} = \tilde{e}_{ijk} + \tilde{f}_{ijk}$, $\tilde{k}_{ij} = \tilde{e}_{ij} + \tilde{f}_{ij}$ given in appendix B we find that the coupling constant must be $a = -\frac{1}{2}$. The spinorial generators are then given by

$$\begin{array}{ll} k_{ij}: & \frac{1}{2}\gamma^{ij} \\ k_i: & \frac{1}{2}\hat{\gamma}\gamma^i \\ \tilde{k}_{ijk}: & -\frac{1}{2}\gamma^{ijk} \\ \tilde{k}_{ij}: & -\frac{1}{2}\hat{\gamma}\gamma^{ij} \end{array} \tag{5.18}$$

(we define $k_{ij} = -k_{ji} = -e_{ji} - f_{ji}$ for $i > j$). We have recovered the well known feature that the spinorial representation of so(9) is the vector representation of so(16) (see [20] for more on this).

We see also that $E_{8(8)}$-invariance forces one to introduce the covariant Dirac operator

$$\gamma^\mu D_\mu \psi = \gamma^\mu (\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma^{ab})\psi + \frac{1}{2.4!}F_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}\psi \tag{5.19}$$

for the Dirac field. This is exactly the same which appears in $D = 11$ supergravity, but it is obtained in that context from supersymmetry.

5.3 IIB

The oxidation of the $E_{8(8)}/SO(16)$ coset theory has another endpoint, in $D = 10$: the bosonic sector of type *IIB* supergravity. There is no manifestly covariant Lagrangian attached to this theory. Indeed, the theory contains a selfdual 4-form, which has no simple (quadratic) manifestly covariant Lagrangian (although it does admit a quadratic non manifestly covariant Lagrangian [28], or a non polynomial manifestly covariant Lagrangian [29]). In spite of the absence of a covariant Lagrangian, the equations of motion are covariant and one may address the following question: is there a "covariant Dirac operator" for fermions in $D = 10$ which reduces to the same $SO(16)$ covariant derivative in $D = 3$?

Following the notations of [25], we have for this theory, in addition to the metric, a dilaton ϕ, an other scalar χ, two 2-forms $A^1_{(2)}$ and $A^2_{(2)}$ with field strength $F^1_{(3)}$ and $F^2_{(3)}$, and a 4-form $B_{(4)}$ with selfdual field strength $H_{(5)}$.

If it exists, the $D = 10$ "covariant Dirac operator" would have the form

$$\begin{aligned} \gamma^\mu \nabla_\mu = \gamma^\mu \partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma^\mu\gamma^{ab} - e^\phi \partial_\mu \chi (a + \tilde{a}\gamma)\gamma^\mu - \frac{1}{3!}e^{\frac{1}{2}\phi}F^1_{\mu\nu\rho}(b + \tilde{b}\gamma)\gamma^{\mu\nu\rho} \\ -\frac{1}{3!}e^{-\frac{1}{2}\phi}F^2_{\mu\nu\rho}(c + \tilde{c}\gamma)\gamma^{\mu\nu\rho} - \frac{1}{5!}H_{\mu\nu\rho\sigma\tau} f \gamma^{\mu\nu\rho\sigma\tau} \ . \end{aligned} \tag{5.20}$$

$\gamma = \gamma^{11}$ is the product of the ten γ^i matrices. As $H_{(5)}$ is selfdual, the dual term

$$H_{\mu\nu\rho\sigma\tau}\gamma\gamma^{\mu\nu\rho\sigma\tau} = (*H)_{\mu\nu\rho\sigma\tau}\gamma^{\mu\nu\rho\sigma\tau} \tag{5.21}$$

is already taken into account. The powers of the dilaton are fixed so that the field strength give the expected fields in $D = 3$.

Now, the axion term $e^\phi \partial_\mu \chi$ is the connection for the $SO(2)$-subgroup of the $SL(2)$ symmetry present in 10 dimensions. Under $SO(2)$-duality, the two two-forms rotate into each other. So, the commutator of the generator $(a + \tilde{a}\gamma)$ multiplying the connection $e^\phi \partial_\mu \chi$ with the generators $(b + \tilde{b}\gamma)\gamma^{\nu\rho}$ multiplying the connection $e^{\frac{1}{2}\phi}F^1_{\mu\nu\rho}$ should reproduce the generator $(c + \tilde{c}\gamma)\gamma^{\nu\rho}$ multiplying the connection $e^{\frac{1}{2}\phi}F^2_{\mu\nu\rho}$. But one has $[(a + \tilde{a}\gamma), (b + \tilde{b}\gamma)\gamma^{\nu\rho}] = 0$, leading to a contradiction.

The problem just described comes from the fact that we have taken a single Dirac fermion. Had we taken instead two Weyl fermions, as it is actually the case for type IIB supergravity, and assumed that they transformed appropriately into each other under the $SO(2)$-subgroup of the $SL(2)$ symmetry, we could have constructed an appropriate covariant derivative. This covariant derivative is in fact given in [30], to which we refer the reader. The $SO(2)$ transformations rules of the spinors — as well as the fact that they must have same chirality in order to transform indeed non trivially into each other — follow from E_8-covariance in 3 dimensions.

5.4 E_7 case

The $E_{7(7)}$ exceptional group is a subgroup of $E_{8(8)}$. As a consequence, the $D=3$ coset $E_{7(7)}/SU(8)$ can be seen as a truncation of the $E_{8(8)}/SO(16)$ coset theory. In fact, this truncation can be made in higher dimension [4]. One can truncate the $D=9$ reduction of the gravity + 3-form theory considered in the last section. If one does not worry about Lagrangian, one can go one dimension higher and view the theory as the truncation of the bosonic sector of type *IIB* supergravity in which one keeps only the vielbein and the chiral 4-form.

In $D=9$, the coupling to fermions obtained in section 5.2 is truncated in a natural way: the components of the covariant Dirac operator acting on fermions are the various fields of the theory, so some of them just disappear with the truncation. The symmetry of the reduced $D=3$ theory is thus preserved: the fermions are coupled to the bosonic fields through a $SU(8)$ covariant derivative, the truncation of the $SO(16)$ covariant derivative of the E_8 case.

The question is about oxidation to $D=10$. Can we obtain this truncated covariant Dirac operator from a covariant Dirac operator of the $D=10$ theory? For the reasons already exposed, if it exists, this operator would act on Dirac fermions as

$$\gamma^\mu\nabla_\mu = \gamma^\mu\partial_\mu - \frac{1}{4}\omega_\mu{}^{ab}\gamma^\mu\gamma^{ab} - a\frac{1}{5!}H_{\mu\nu\rho\sigma\tau}\gamma^{\mu\nu\rho\sigma\tau} \tag{5.22}$$

where we have denoted by H the selfdual field strength.

With notations analogous to the E_8 case, we can write the $D=3$ reduction of the covariant Dirac operator as

$$\begin{aligned}\gamma^\mu\nabla_\mu = \gamma^\mu\partial_\mu - \frac{1}{4}\tilde{\omega}_\mu{}^{ab}\gamma^\mu\gamma^{ab} - \frac{1}{4}e^{\frac{1}{2}\vec{b}_i.\vec{\phi}}\mathcal{F}_{(2)\mu\nu i}\gamma^{\mu\nu}\gamma^i - \frac{1}{4}e^{\frac{1}{2}\vec{b}_{ij}.\vec{\phi}}\mathcal{F}_{(1)\mu ij}\gamma^\mu\gamma^{ij}\\ - a\frac{1}{2.3!}e^{\frac{1}{2}\vec{a}_{ijk}.\vec{\phi}}H_{(2)\mu\nu ijk}\gamma^{\mu\nu}\gamma^{ijk} - a\frac{1}{4!}e^{\frac{1}{2}\vec{a}_{ijkl}.\vec{\phi}}H_{(1)\mu ijkl}\gamma^\mu\gamma^{ijkl}\ .\end{aligned} \tag{5.23}$$

Because of the selfduality of H, the 2-forms $H_{(2)}$ and the 1-forms $H_{(1)}$ are in fact dual. Using $\gamma=\gamma^0\gamma^1\ldots\gamma^9$, the covariant Dirac operator turns into

$$\begin{aligned}\gamma^\mu\nabla_\mu = \gamma^\mu\partial_\mu + \frac{1}{4}\tilde{\omega}_\mu{}^{ab}\gamma^\mu\gamma^{ab} + \frac{1}{2}e^{-\frac{1}{2}\vec{b}_i.\vec{\phi}}\mathcal{G}_{(1)\mu i}\gamma^\mu\hat{\gamma}\gamma^i + \frac{1}{2.2}e^{\frac{1}{2}\vec{b}_{ij}.\vec{\phi}}\mathcal{F}_{(1)\mu ij}\gamma^\mu\gamma^{ij}\\ + a\frac{1}{4!}e^{\frac{1}{2}\vec{a}_{ijkl}.\vec{\phi}}H_{(1)\mu ijkl}\gamma^\mu(1+\gamma)\gamma^{ijkl}\ .\end{aligned} \tag{5.24}$$

The embedding of $E_{7(7)}/SU(8)$ in $E_{8(8)}/SO(16)$ gives the following identifications:

$$\begin{aligned}H_{(1)1ijk} &= F_{(1)(i+1)(j+1)(k+1)}\\ H_{(1)ijkl} &= -\frac{1}{2}\epsilon^{12(i+1)(j+1)(k+1)(l+1)mn}G_{(1)mn}\\ \mathcal{F}_{(1)1i} &= F_{(1)12(i+1)}\\ \mathcal{G}_{(1)1} &= -G_{(2)12}\\ \mathcal{F}_{(1)ij} &= \mathcal{F}_{(1)(i+1)(j+1)}\\ \mathcal{G}_{(1)i} &= \mathcal{G}_{(1)(i+1)}\end{aligned} \tag{5.25}$$

with $2 \leq i,j,k,l \leq 7$. We thus have to check that the matrices in (5.24) form the following representation:

$$\begin{array}{rcl} a(1+\gamma)\gamma^{1ijk} & \sim & \tilde{k}_{(i+1)(j+1)(k+1)} \\ a(1+\gamma)\gamma^{ijkl} & \sim & \frac{1}{2}\epsilon^{12(i+1)(j+1)(k+1)(l+1)mn}\tilde{k}_{mn} \\ \frac{1}{2}\gamma^{1i} & \sim & \tilde{k}_{12(i+1)} \\ \frac{1}{2}\hat{\gamma}\gamma^{1} & \sim & -\tilde{k}_{12} \\ \frac{1}{2}\gamma^{ij} & \sim & k_{(i+1)(j+1)} \\ \frac{1}{2}\hat{\gamma}\gamma^{i} & \sim & k_{(i+1)} \ . \end{array} \tag{5.26}$$

This is true if and only if

$$-4a^2(1+\gamma) = \frac{1}{2} \ . \tag{5.27}$$

This has to be understood as an identity between operators acting on fermions. In fact, this means that we must restrict to Weyl spinors, with $\gamma = +1$ when acting on them. Due to the even number of γ matrices involved in all generators in (5.26), the su(8) algebra preserves the chirality of spinors. We get in addition the value of the coupling constant:

$$a = \pm\frac{i}{4} \ . \tag{5.28}$$

5.5 E_6 case

The E_6 case is more simple. One has a Lagrangian in all dimensions. In dimension 3, the scalar coset is $E_{6(6)}/Sp(4)$. Maximal oxidation is a $D = 8$ theory with a 3-form, a dilaton and an axion (scalar) [4]. It can be seen as a truncation of the E_8 case in all dimensions. In the compact subalgebra of so(16) given in (B.5), one should remove generators with one or two indices in $\{1,2,3\}$ while keeping $\tilde{k}_{123}$.

In fact, all the matrices involved in the Dirac representation can be expressed in terms of a $D = 8$ Clifford algebra. For most generators, it is trivial to check that they involve only γ^i matrices with $i \neq 1,2,3$. The single nontrivial case is $\tilde{k}_{123}$ which is represented by $-\frac{1}{2}\gamma^{123}$ in the eleven-dimensional Clifford algebra. But from the fact that $\gamma^{(10)} = \gamma^0\gamma^1\ldots\gamma^9$, we can write γ^{123} as the product of all other γ matrices: $\gamma^{123} = \gamma^{0456789(10)}$. As a consequence, the $D = 8$ Clifford algebra is sufficient to couple a Dirac fermion to this model: we can couple a single $D = 8$ Dirac fermion.

6 G_2 case

6.1 Bosonic sector

Let us consider the Einstein-Maxwell system in $D = 5$, with the FFA term prescribed by supersymmetry [16],

$$\mathcal{L}_5 = R*\mathbb{1} - \tfrac{1}{2}{*F_{(2)}} \wedge F_{(2)} + \tfrac{1}{3\sqrt{3}}F_{(2)} \wedge F_{(2)} \wedge A_{(1)} \ . \tag{6.1}$$

This action is known to be relevant to G_2 [31, 4]. Upon reduction to $D = 3$, the Lagrangian is [4]

$$\begin{aligned} \mathcal{L} \ = \ & R*\mathbb{1} - \tfrac{1}{2}{*d\vec{\phi}} \wedge d\vec{\phi} - \tfrac{1}{2}e^{\phi_2 - \sqrt{3}\phi_1} *\mathcal{F}^1_{(1)2} \wedge \mathcal{F}^1_{(1)2} - \tfrac{1}{2}e^{\frac{2}{\sqrt{3}}\phi_1} *F_{(1)1} \wedge F_{(1)1} \\ & -\tfrac{1}{2}e^{\phi_2 - \frac{1}{\sqrt{3}}\phi_1} *F_{(1)2} \wedge F_{(1)2} - \tfrac{1}{2}e^{-\phi_2 - \sqrt{3}\phi_1} *\mathcal{F}^1_{(2)} \wedge \mathcal{F}^1_{(2)} \\ & -\tfrac{1}{2}e^{-2\phi_2} *\mathcal{F}^2_{(2)} \wedge \mathcal{F}^2_{(2)} - \tfrac{1}{2}e^{-\phi_2 - \frac{1}{\sqrt{3}}\phi_1} *F_{(2)} \wedge F_{(2)} + \tfrac{2}{\sqrt{3}}dA_{(0)1} \wedge dA_{(0)2} \wedge A_{(1)} \ . \end{aligned} \tag{6.2}$$

After dualising the vector potentials to give axions, there will be six axions, together with the two dilatons. The dilaton vectors $\vec{\alpha}_1 = (-\sqrt{3}, 1)$ and $\vec{\alpha}_2 = (\frac{2}{\sqrt{3}}, 0)$, corresponding to the axions $\mathcal{A}^1_{(0)2}$ and $A_{(0)1}$, are the simple roots of G_2, with the remaining dilaton vectors expressed in terms of these as

$$(-\tfrac{1}{\sqrt{3}}, 1) = \vec{\alpha}_1 + \vec{\alpha}_2 \ , \quad (\tfrac{1}{\sqrt{3}}, 1) = \vec{\alpha}_1 + 2\vec{\alpha}_2 \ , \quad (\sqrt{3}, 1) = \vec{\alpha}_1 + 3\vec{\alpha}_2 \ , \quad (0, 2) = 2\vec{\alpha}_1 + 3\vec{a}_3 \ . \tag{6.3}$$

The resulting $D = 3$ lagrangian is a $G_2/SO(4)$ σ-model coupled to gravity. The field strength of this σ-model is

$$\begin{aligned} \mathcal{G} = \frac{1}{2}\mathrm{d}\vec{\phi}.\vec{H} + e^{\frac{1}{2}\vec{\alpha}_1.\vec{\phi}}\mathcal{F}_{(1)}{}^1{}_2\epsilon_1 + e^{\frac{1}{2}(\vec{\alpha}_1+3\vec{\alpha}_2).\vec{\phi}}\mathcal{G}_{(1)}{}^1\epsilon_5 + e^{\frac{1}{2}(2\vec{\alpha}_1+3\vec{\alpha}_2).\vec{\phi}}\mathcal{G}_{(1)}{}^2\epsilon_6 \\ +e^{\frac{1}{2}\vec{\alpha}_2.\vec{\phi}}F_{(1)1}\epsilon_2 + e^{\frac{1}{2}(\vec{\alpha}_1+\vec{\alpha}_2).\vec{\phi}}F_{(1)2}\epsilon_3 + e^{-\frac{1}{2}(\vec{\alpha}_1+2\vec{\alpha}_2).\vec{\phi}}G_{(1)}\epsilon_4 \ . \end{aligned} \tag{6.4}$$

where $G_{(1)}$ is the dual of F_2, and the notation ϵ_1, ϵ_2, ϵ_3, ϵ_4, ϵ_5 and ϵ_6 is explained in the appendix.

6.2 Fermions

We want to add Dirac fermion in $D = 5$, with a coupling which in the $D = 3$ reduction is covariant with respect to $SO(1, 2) \times SO(4)$. From what we have already learned, this should be possible, with a representation which is trivial on one of the two $SU(2)$ factors of $SO(4) \simeq (SU(2) \times SU(2))/\mathbb{Z}_2$, since we have already seen in the analysis of the gravitational sector that the Clifford algebra contains $SO(1, 2) \times SU(2)$ representations.

To check if we can indeed derive such a representation from a consistent $D = 5$ coupling, we add to the lagrangian 6.1 a Dirac fermion with a Pauli coupling,

$$\mathcal{L}_\psi = \sqrt{-g}\bar{\psi}(\gamma^\mu\partial_\mu - \frac{1}{4}\omega_{\mu,mn}\gamma^{mn} - \frac{1}{2}\alpha\gamma^{\mu\nu}F_{(2)\ \mu\nu})\psi \tag{6.5}$$

where α is a coupling constant which will be determined below. Upon toroidal reduction to $D = 3$, the last term of (6.5) becomes,

$$\frac{\alpha}{2}\sqrt{-\hat{g}}\bar{\hat{\psi}}(e^{-\frac{1}{2}(\vec{\alpha}_1+2\vec{\alpha}_2)\cdot\vec{\phi}}\gamma^{ab}F_{(2)\ ab} + 2e^{\frac{1}{2}\vec{\alpha}_2\cdot\vec{\phi}}\gamma^a\gamma^1F_{(1)1\ a} + 2e^{\frac{1}{2}(\vec{\alpha}_1+\vec{\alpha}_2)\cdot\vec{\phi}}\gamma^a\gamma^2F_{(1)2\ a})\hat{\psi} \tag{6.6}$$

Let us dualize the 2 form field strengths. We get for the dimensional reduction of the whole lagrangian (6.5),

$$\begin{aligned} \mathcal{L}^{(3)}_\psi \ = \ & \sqrt{-\hat{g}}\bar{\hat{\psi}}\gamma_c(\gamma^\mu\partial_\mu - \frac{1}{4}\hat{\omega}_{\mu,mn}\gamma^{mn} \\ & -\frac{1}{2}e^{\frac{1}{2}(\vec{\alpha}_1+2\vec{\alpha}_2)\cdot\vec{\phi}}\Gamma_4G^c - \frac{1}{2}e^{\frac{1}{2}\vec{\alpha}_2\cdot\vec{\phi}}\Gamma_2F^c_{(1)1} - \frac{1}{2}e^{\frac{1}{2}(\vec{\alpha}_1+\vec{\alpha}_2)\cdot\vec{\phi}}\Gamma_3F^c_{(1)2} \\ & -\frac{1}{2}e^{\frac{1}{2}(\vec{\alpha}_1+3\vec{\alpha}_2).\vec{\phi}}\Gamma_5\mathcal{G}^c_1 - \frac{1}{2}e^{\frac{1}{2}(2\vec{\alpha}_1+3\vec{\alpha}_2).\vec{\phi}}\Gamma_6\mathcal{G}^c_2 - \frac{1}{2}e^{\frac{1}{2}\vec{\alpha}_1.\vec{\phi}}\Gamma_1\mathcal{F}^{c\ 1}_{(1)\ \ 2})\hat{\psi} \end{aligned} \tag{6.7}$$

where $\Gamma_1 = \frac{1}{2}\gamma^{12}$, $\Gamma_2 = 2\alpha\gamma^1$, $\Gamma_3 = 2\alpha\gamma^2$, $\Gamma_4 = 2\alpha\hat{\gamma}$, $\Gamma_5 = \frac{1}{2}\hat{\gamma}\gamma^1$ and $\Gamma_6 = \frac{1}{2}\hat{\gamma}\gamma^2$. Notice that $\hat{\gamma} = -i\gamma^{12}$ because the product of all gamma matrices $\gamma^0\gamma^1\gamma^2\gamma^3\gamma^4 = \hat{\gamma}\gamma^1\gamma^2$ in $D = 5$ can be equated to i.

As in the case of the other algebras encountered above, we need to check that the Γ_i's obey the commutation relations of the maximally compact subalgebra of G_2, i.e., obey the same commutation relations as the k_i's, given in appendix C. We find that the commutation relations are indeed fulfilled provided we take $\alpha = ia$, with a solution of the quadratic equation $16a^2 + \frac{8}{\sqrt{3}}a - 1 = 0$, which implies $\alpha = -i\frac{\sqrt{3}}{4}$ or $\alpha = \frac{i}{4\sqrt{3}}$. The two different solutions correspond to a non trivial representation for either the left or the right factor $SU(2)$. Thus, we see again

that the fermions are compatible with G_2-invariance and we are led to introduce the covariant Dirac operator

$$\gamma^\mu D_\mu \psi = \gamma^\mu (\partial_\mu - \frac{1}{4}\omega_{\mu,mn}\gamma^{mn})\psi - \frac{1}{2}\alpha\gamma^{\mu\rho\sigma}F_{(2)\ \rho\sigma}\psi \tag{6.8}$$

(with α equal to one of the above values) for the spin-1/2 field. This is the same expression as the one that followed from supersymmetry [16].

Another approach of this problem is to remember that G_2 can be embedded in $D_4 = SO(4,4)$ [4]. The maximal oxidation is $D = 6$ and contains a 2-form in addition to gravity. After reduction on a circle, we get two dilatons and three 1-forms: the original 2-form and its Hodge dual both reduce to 1-forms, and we have also the Kaluza-Klein 1-form. The model we are dealing with is obtained by equating these three 1-forms, and setting the dilatons to zero [4]. It is clear that this projection do respect the covariance of the fermionic coupling obtained by reduction of (4.8). In $D = 3$, the compact gauge group is projected from $SO(4)\times SO(4)$ onto $SO(4)$, in addition to the unbroken $SO(1,2)$. All other terms in the connection are indeed set to zero by the embedding. The $D = 6$ spinor can be chosen to have a definite chirality. Each chirality corresponds to a different choice of α after dimensional reduction.

7 Non-simply laced algebras B_n, C_n, F_4

All the non-simply laced algebras can be embedded in simply laced algebras [4]. Therefore, we can find the appropriate coupling by taking the one obtained for the simply laced algebras and by performing the same identifications as for the bosonic sector.

$B_n = SO(n,n+1)$ (with maximal compact subgroup $SO(n) \times SO(n+1)$) can be obtained from D_{n+1} by modding out the $\mathbb{Z}_2$ symmetry of the diagram. As the D_{n+1} coset can be oxidised up to $D = n+3$, we must consider a $D = n+3$ Clifford algebra. The B_n coset has its maximal oxidation in one dimension lower. If n is even, $D = n+3$ and $D = n+2$ Dirac spinors are the same: the embedding gives a coupling to a single $D = n+2$ Dirac spinor. Is n is odd, this argument is no longer sufficient. However, due to the fact that all elements of the compact subalgebra of $SO(n+1, n+1)$ are represented by a product of an even number of gamma matrices, we can take a Weyl spinor in $D = n+3$: it gives a single Dirac spinor in $D = n+2$. It is thus possible couple the maximal oxidation of the B_n coset to a single Dirac spinor, such that it reduces to a Dirac coupling in $D = 3$, covariant with respect to $SO(1,2)\times SO(n)\times SO(n+1)$. We leave the details to the reader.

For $C_n = Sp(n)$ (with maximal compact subgroup $U(n)$), the maximal oxidation lives in $D = 4$. The embedding in A_{2n-1} couples the bosonic degrees of freedom to a $D = 2n+2$ spinor. As it is an even dimension, the Weyl condition can be again imposed, so that we get a $D = 2n$ Dirac spinor. It is not possible to reduce further the number of components: the representation involves product of odd numbers of gamma matrices. In $D = 4$, this gives a coupling to 2^{n-2} Dirac spinors.

The situation for F_4 is similar, when considering the embedding in E_6. The E_6 coset can be oxidised up to $D = 8$, with a consistent fermionic coupling to a Dirac spinor. As the coupling to the 3-form involves the product of 3 gamma matrices, it is not possible to impose the Weyl condition. The maximal oxidation of the $F_4/(SU(2) \times Sp(3))$ coset, which lives in dimension 6, is thus coupled to a pair of Dirac spinors.

For C_n and F_4, the embeddings just described give a coupling to respectively 2^{n-2} and 2 Dirac spinors in the maximally oxidised theory. We have not investigated in detail whether one could construct invariant theories with a smaller number of spinors.

8 G^{++} Symmetry

The somewhat magic emergence of unexpected symmetries in the dimensional reduction of gravitational theories has raised the question of whether these symmetries, described by the algebra G in three dimensions, are present prior to reduction or are instead related to toroidal compactification. It has been argued recently that the symmetries are, in fact, already present in the maximally oxidized version of the theory (see [32] for early work on the E_8-case) and are part of a much bigger, infinite-dimensional symmetry, which could be the overextended algebra G^{++} [6, 7, 8], the very extended algebra G^{+++} [11, 12], or a Borcherds superalgebra related to it [13]. There are different indications that this should be the case, including a study of the BKL limit of the dynamics [21], which leads to "cosmological billiards" [22].

In [10], an attempt was made to make the symmetry manifest in the maximal oxidation dimension by reformulating the system as a $(1+0)$-non linear sigma model $G^{++}/K(G^{++})$. The explicit case of E_{10} was considered. It was shown that at low levels, the equations of motion of the bosonic sector of 11-dimensional supergravity can be mapped on the equations of motion of the non linear sigma model $E_{10}/K(E_{10})$. The matching works for fields associated with roots of E_{10} whose height does not exceed 30 (see also [33]).

We now show that this matching works also for Dirac spinors. We consider again the explicit case of E_{10} for definiteness. We show that the Dirac Lagrangian for a Dirac spinor in eleven dimensions, coupled to the supergravity three-form as in section 5.2, is covariant under $K(E_{10})$, at least up to the level where the bosonic matching is successful. [For related work on including fermions in these infinite-dimensional algebras, see [34].]

Our starting point is the action for the non linear sigma model $E_{10}/K(E_{10})$ in 1+0 dimension coupled to Dirac fermions transforming in a representation of $K(E_{10})$. We follow the notations of [22]. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2} n^{-1} < \mathcal{P}|\mathcal{P} > + i\Psi^\dagger D_t \Psi \tag{8.1}$$

where we have introduced a lapse function n to take into account reparametrization invariance in time. The $K(E_{10})$ connection is

$$\mathcal{Q} = \sum_{\alpha\in\Delta_+} \sum_{s=1}^{mult(\alpha)} \mathcal{Q}_{\alpha,s} K_{\alpha,s} \tag{8.2}$$

while the covariant derivative is

$$D_t\Psi = \dot{\Psi} - \sum_{\alpha,s} \mathcal{Q}_{\alpha,s} T_{\alpha,s}\Psi \tag{8.3}$$

where the $T_{\alpha,s}$ are the generators of the representation in which Ψ transforms (there is an infinity of components for Ψ).

In the Borel gauge, the fermionic part of the Lagrangian becomes

$$i\Psi^\dagger\dot{\Psi} - \frac{i}{2}\sum_{\alpha,s} e^{\alpha(\beta)} j_{\alpha,s}\Psi^\dagger T_{\alpha,s}\Psi \tag{8.4}$$

where β^μ are now the Cartan subalgebra variables (i.e., we parametrize the elements of the Cartan subgoup as $\exp(\beta^\mu h_\mu)$) and $\alpha(\beta)$ the positives roots. The "currents" $j_{\alpha,s}$ (denoted by $\mathcal{F}_{\alpha,s}$ in previous sections) are, as before, the coefficients of the positive generators in the expansion of the algebra element $\dot{\mathcal{V}}\mathcal{V}^{-1}$,

$$\dot{\mathcal{V}}\mathcal{V}^{-1} = \dot{\beta}^\mu h_\mu + \sum_{\alpha\in\Delta_+} \sum_{s=1}^{mult(\alpha)} \exp\left(\alpha(\beta)\right) j_{\alpha,s} E_{\alpha,s} \tag{8.5}$$

We must compare (8.4) with the Dirac Lagrangian in 11 dimensions with coupling to the 3-form requested by E_8 invariance,

$$e\overline{\psi}\left(\gamma^\mu\partial_\mu - \frac{1}{4}\omega_{\mu\ ab}\gamma^\mu\gamma^{ab} - \frac{1}{2.4!}F_{\mu\nu\rho\sigma}\gamma^{\mu\nu\rho\sigma}\right)\psi \tag{8.6}$$

where e is now the determinant of the space-time vielbein. To make the comparison easier, we first take the lapse n equal to one (standard lapse N equal to e^{-1}) since both (8.4) and (8.6) are reparametrization invariant in time. We further split the Dirac Lagrangian (8.6) into space and time using a zero shift ($N^k = 0$) and taking the so-called time gauge for the vielbeins e^a_μ, namely no mixed space-time component. This yields

$$\begin{aligned} i\chi^\dagger\left(\dot\chi - \frac{1}{4}\omega^R_{ab}\gamma^{ab}\chi - \frac{1}{2.3!}F_{0abc}\gamma^{abc}\chi - \frac{e}{2.4!\,6!}\varepsilon_{abcdp_1p_2\cdots p_6}F^{abcd}\gamma^{p_1\cdots p_6}\chi\right) \\ +i\chi^\dagger\left(-\frac{e}{2.2!\,8!}\omega_k{}^{ab}\varepsilon_{abp_1\cdots p_8}\gamma^k\gamma^{p_1\cdots p_8}\chi + \frac{e}{10!}\varepsilon_{p_1\cdots p_{10}}\gamma^k\gamma^{p_1\cdots p_{10}}\partial_k\chi\right) \end{aligned} \tag{8.7}$$

where e is now the determinant of the spatial vielbein and where the Dirac field is taken to be Majorana (although this is not crucial) and has been rescaled as $\chi = e^{1/2}\psi$. In (8.7), the term ω^R_{ab} stands for

$$\omega^R_{ab} = -\frac{1}{2}(e_a{}^\mu\dot e_{\mu b} - e_b{}^\mu\dot e_{\mu a}) \tag{8.8}$$

A major difference between (8.4) and (8.7) is that Ψ has an infinite number of components while χ has only 32 components. But Ψ depends only on t, while χ is a spacetime field. We shall thus assume, in the spirit of [10], that Ψ collects the values of χ and its successive spatial derivatives at a given spatial point,

$$\Psi^\dagger = (\chi^\dagger, \partial_k\chi^\dagger, \cdots) \tag{8.9}$$

[The dictionary between Ψ on the one hand and χ and its successive derivatives on the other hand might be more involved (the derivatives might have to be taken in privileged frames and augmented by appropriate corrections) but this will not be of direct concern for us here. We shall loosely refer hereafter to the "spatial derivatives of χ" for the appropriate required modifications.] We are thus making the strong assumption that by collecting χ and its derivatives in a single infinite dimensional object, one gets a representation of $K(E_{10})$. It is of course intricate to check this assertion, partly because $K(E_{10})$ is poorly understood [34]. Our only justification is that it makes sense at low levels.

Indeed, by using the bosonic, low level, dictionary of [10], we do see the correct connection terms appearing in (8.7) at levels 0 (ω^R_{ab} term), 1 (electric field term) and 2 (magnetic field term). The corresponding generators $\gamma^{a_1a_2}$, $\gamma^{a_1a_2a_3}$ and $\gamma^{a_1\cdots a_6}$ do reproduce the low level commutation relations of $K(E_{10})$.

The matching between the supergravity bosonic equations of motion and the nonlinear sigma model equations of motion described in [10] goes slightly beyond level 2 and works also for some roots at level 3. We shall refer to this as "level 3^-". To gain insight into the matching at level 3^- for the fermions, we proceed as in [10] and consider the equations in the homogeneous context of Bianchi cosmologies [35, 36] (see also [37]). The derivative term $\partial_k\chi$ then drops out — we shall have anyway nothing to say about it here, where we want to focus on the spin connection term $\omega_k{}^{ab}$. In the homogeneous context, the spin connection term becomes

$$\omega_{abc} = \frac{1}{2}\left(C_{cab} + C_{bca} - C_{abc}\right) \tag{8.10}$$

in terms of the structure constants $C^a{}_{bc} = -C^a{}_{cb}$ of the homogeneity group expressed in homogeneous orthonormal frames (the $C^a{}_{bc}$ may depend on time). We assume that the traces $C^a{}_{ac}$

vanish since these correspond to higher height and go beyond the matching of [10], i.e., beyond level 3^-. In that case, one may replace ω_{abc} by $(1/2)C_{abc}$ in (8.7) as can be seen by using the relation

$$\varepsilon_{abp_1\cdots p_8}\gamma^k\gamma^{p_1\cdots p_8} = \varepsilon_{abp_1\cdots p_8}\gamma^{kp_1\cdots p_8} + \varepsilon_{abkp_2\cdots p_8}\gamma^{p_2\cdots p_8}.$$

The first term drops from (8.7) because ${\omega^a}_{ba} = 0$, while the second term is completely antisymmetric in a, b, k. Once ω_{abc} is replaced by $(1/2)C_{abc}$, one sees that the remaining connection terms in (8.7), i.e., the one involving a product of nine γ-matrices, agree with the dictionary of [10]. Furthermore, the corresponding level three generators $\gamma^c\gamma^{p_1\cdots p_8}$ also fulfill the correct commutation relations of $K(E_{10})$ up to the requested order.

To a large extent, the E_{10} compatibility of the Dirac fermions up to level 3^- exhibited here is not too surprising, since it can be viewed as a consequence of SL_{10} covariance (which is manifest) and the hidden E_8 symmetry, which has been exhibited in previous sections. The real challenge is to go beyond level 3^- and see the higher positive roots emerge on the supergravity side. These higher roots might be connected, in fact, to quantum corrections [38] or higher spin degrees of freedom.

9 BKL limit

We investigate in this final section how the Dirac field modifies the BKL behaviour. To that end, we first rewrite the Lagrangian (8.1) in Hamiltonian form. The fermionic part of the Lagrangian is already in first order form (with $i\Psi^\dagger$ conjugate to Ψ), so we only need to focus on the bosonic part. The conjugate momenta to the Cartan fields β^μ are unchanged in the presence of the fermions since the time derivatives $\dot\beta^\mu$ do not appear in the connection $\mathcal{Q}_{\alpha,s}$. However, the conjugate momenta to the off-diagonal variables parameterizing the coset do get modified. How this affects the Hamiltonian is easy to work out because the time derivatives of these off-diagonal group variables occur linearly in the Dirac Lagrangian, so the mere effect of the Dirac term is to shift their original momenta. Explicitly, in terms of the (non-canonical) momentum-like variables

$$\Pi_{\alpha,s} = \frac{\delta\mathcal{L}}{\delta j_{\alpha,s}} \tag{9.1}$$

introduced in [39, 22], one finds

$$\Pi_{\alpha,s} = \Pi^{old}_{\alpha,s} - \frac{1}{2}\exp\left(\alpha(\beta)\right) J^F_{\alpha,s} \tag{9.2}$$

where $\Pi^{old}_{\alpha,s}$ is the bosonic contribution (in the absence of fermions) and where $J^F_{\alpha,s}$ are the components of the fermionic $K(G^{++})$-current, defined by

$$J^F_{\alpha,s} = i\Psi^\dagger T_{\alpha,s}\Psi. \tag{9.3}$$

The currents $J^F_{\alpha,s}$ are real.

It follows that the Hamiltonian associated with (8.1) takes the form

$$\mathcal{H} = n\left(\frac{1}{2}G^{\mu\nu}\pi_\mu\pi_\nu + \sum_{\alpha\in\Delta_+}\sum_{s=1}^{mult(\alpha)}\exp\left(-2\alpha(\beta)\right)\left(\Pi_{\alpha,s} + \frac{1}{2}\exp\left(\alpha(\beta)\right)J^F_{\alpha,s}\right)^2\right) \tag{9.4}$$

If one expands the Hamiltonian, one gets

$$\mathcal{H} = n\left(\frac{1}{2}G^{\mu\nu}\pi_\mu\pi_\nu + \sum_{\alpha,s}\exp\left(-2\alpha(\beta)\right)\Pi^2_{\alpha,s} + \sum_{\alpha,s}\exp\left(-\alpha(\beta)\right)\Pi_{\alpha,s}J^F_{\alpha,s} + \frac{1}{4}C\right) \tag{9.5}$$

where C is (up to a numerical factor) the quadratic Casimir of the fermionic representation,

$$C = \sum_{\alpha,s} (J^F_{\alpha,s})^2. \tag{9.6}$$

We see that, just as in the pure bosonic case, the exponentials involve only the positive roots with negative coefficients. However, we obtain, in addition to the bosonic walls, also their square roots. All the exponentials in the Hamiltonian are of the form $\exp(-2\rho(\beta))$, where $\rho(\beta)$ are the positive roots or half the positive roots.

In order to investigate the asymptotic BKL limit $\beta^\mu \to \infty$, we shall treat the $K(G^{++})$-currents as classical real numbers and consider their equations of motion that follow from the above Hamiltonian, noting that their Poisson brackets $[J^F_{\alpha,s}, J^F_{\alpha',s'}]$ reproduce the $K(G^{++})$-algebra. This is possible because the Hamiltonian in the Borel gauge involves only the Ψ-field through the currents. This is a rather remarkable property. [A "classical" treatment of fermions is well known to be rather delicate. One can regard the dynamical variables, bosonic and fermionic, as living in a Grassmann algebra. In that case, bilinear in fermions are "pure souls" and do not influence the behaviour of the "body" parts of the group elements, which are thus trivially governed by the same equations of motion as in the absence of fermions. However, it is reasonable to expect that the currents $J^F_{\alpha,s}$ have a non trivial classical limit (they may develop non-vanishing expectation values) and one might treat them therefore as non-vanishing real numbers. This is technically simple here because the currents obey closed equations of motion. It leads to interesting consequences.]

Next we observe that $[J^F_{\alpha,s}, C] = 0$. It follows that the quadratic Casimir C of the fermionic representation is conserved. Furthermore, it does not contribute to the dynamical Hamiltonian equations of motion for the group variables or the currents. By the same reasoning as in [22], one can then argue that the exponentials tend to infinite step theta functions and that all variables except the Cartan ones, i.e., the off-diagonal group variables and the fermionic currents, asymptotically freeze in the BKL limit.

Thus, we get the same billiard picture as in the bosonic case, with same linear forms characterizing the walls (some of the exponential walls are the square roots of the bosonic walls). But the free motion is governed now by the Hamiltonian constraint

$$G^{\mu\nu}\pi_\mu\pi_\nu + M^2 \approx 0 \tag{9.7}$$

with $M^2 = C/2 > 0$. This implies that the motion of the billiard ball is timelike instead of being lightlike as in the pure bosonic case. This leads to a non-chaotic behaviour, even in those cases where the bosonic theory is chaotic. Indeed, a timelike motion can miss the walls, even in the hyperbolic case. This is in perfect agreement with the results found in [23] for the four-dimensional theory.

Our analysis has been carried out in the context of the sigma model formulation, which is equivalent to the Einstein-Dirac-p-form system only for low Kac-Moody levels. However, the low levels roots are precisely the only relevant ones in the BKL limit ("dominant walls"). Thus, the analysis applies also in that case. Note that the spin 1/2 field itself does not freeze in the BKL limit, even after rescaling by the quartic root of the determinant of the spatial metric, but asymptotically undergoes instead a constant rotation in the compact subgroup, in the gauge $n = 1$ (together with the Borel gauge). Note also that the same behaviour holds if one adds a mass term to the Dirac Lagrangian, since this term is negligible in the BKL limit, being multiplied by e, which goes to zero.

One might worry that the coefficients $\Pi_{\alpha,s} J^F_{\alpha,s}$ of the square roots of the bosonic walls have no definite sign. This is indeed true but generically of no concern for the following reason: in the region $\alpha(\beta) < 0$ outside the billiard table where the exponential terms are felt and in fact blow

up with time at a given configuration point ($\alpha(\beta) \to -\infty$) [22], the wall $\exp(-2\alpha(\beta))$ dominates the wall $\exp(-\alpha(\beta))$ coming from the fermion and the total contribution is thus positive. The ball is repelled towards the billiard table.

10 Conclusions

In this paper, we have shown that the Dirac field is compatible with the hidden symmetries that emerge upon toroidal dimensional reduction to three dimensions, provided one appropriately fix its Pauli couplings to the p-forms. We have considered only the split real form for the symmetry (duality) group in three dimensions, but similar conclusions appear to apply to the non-split forms (we have verified it for the four-dimensional Einstein-Maxwell-Dirac system, which leads to the $SU(2,1)/S(U(2) \times U(1))$ coset in three dimensions). We have also indicated that the symmetry considerations reproduce some well known features of supersymmetry when supersymmetry is available.

We have also investigated the compatibility of the Dirac field with the conjectured infinite-dimensional symmetry G^{++} and found perfect matching with the non-linear sigma model equations minimally coupled to a $(1+0)$ Dirac field, up to the levels where the bosonic matching works.

Finally, we have argued that the Dirac fermions destroy chaos (when it is present in the bosonic theory), in agreement with the findings of [23]. This has a rather direct group theoretical interpretation (motion in Cartan subalgebra becomes timelike) and might have important implications for the pre-big-bang cosmological scenario and the dynamical crossing of a cosmological singularity [40, 41].

It would be of interest to extend these results to include the spin 3/2 fields, in the supersymmetric context. In particular, 11-dimensional supergravity should be treated. To the extent that E_{10} invariance up to the level 3^- is a mere consequence of E_8 invariance in three dimensions and SL_{10} convariance, one expects no new feature in that respect since the reduction to three dimensions of full supergravity is indeed known to be E_8 invariant [27]. But perhaps additional structure would emerge. Understanding the BKL limit might be more challenging since the spin 3/2 fields might not freeze, even after rescaling.

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A Conventions for D_n

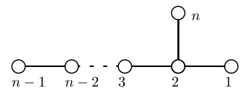
A.1 Generators and Algebra

We can express the positive generators of D_n as follow,

$$
\begin{aligned}
e_{b_{ij}} &\equiv [e_i, [\ldots, [e_{j-2}, e_{j-1}]\ldots] \quad i<j \\
e_{b_i} &\equiv [\tilde{e}_{n-1}, e_{b_{1i}}] \\
e_{a_i} &\equiv [e_{b_{in-1}}, e_{n-1}] \qquad (A.1)\\
e_{a_{ij}} &\equiv [[e_n, e_{b_{2j}}], e_{b_{1i}}] \; i<j \qquad (A.2)
\end{aligned}
$$

where $\tilde{e}_{n-1} = [e_n, [e_{b_{2n-1}}, e_{n-1}]]$ and where $i = 1, \ldots, n-1$ and $e_{b_{ii}}$ must be understood as being absent. The Chevalley-Serre generators of D_n, namely $\{e_m \mid m = 1, \ldots, n\}$, are given by $e_i = e_{b_{ii+1}}$

$(i = 1, ..., n-2)$, $e_{n-1} = e_{a_{n-1}}$ and $e_n = e_{a_{12}}$. These generators are associated to the vertices numbered as shown in the following Dynkin diagram,



Their non vanishing commutation relations are

$$
\begin{aligned}
[e_{b_{ij}}, e_{b_{mn}}] &= \delta_{jm} e_{b_{in}} - \delta_{in} e_{b_{mj}} \\
[e_{b_{ij}}, e_{b_m}] &= -\delta_{im} e_{b_j} \\
[e_{b_{ij}}, e_{a_{mn}}] &= -\delta_{im} e_{a_{jn}} - \delta_{in} e_{a_{mj}} + \delta_{im} e_{a_{nj}} \\
[e_{b_{ij}}, e_{a_m}] &= \delta_{jm} e_{a_i} \\
[e_{a_{ij}}, e_{a_m}] &= -\delta_{im} e_{b_j} + \delta_{jm} e_{b_i}
\end{aligned}
\tag{A.3}
$$

Notations are similar for the negative generators (with f's instead of e's). One easily verifies that the normalization factors N_α are all equal to one, $K(e_\alpha, f_\beta) = -\delta_{\alpha\beta}$.

A.2 Compact subgroup

As explained above, the involution τ is such that $\tau(h_i) = -h_i$, $\tau(e_\alpha) = f_\alpha$ and $\tau(f_\alpha) = e_\alpha$ so that a basis of the maximally compact subalgebra of D_n reads $k_\alpha = e_\alpha + f_\alpha$ where $\alpha = \{a_{ij}, a_i, b_{ij}, b_i\}$ and $i < j = 1, ..., n-1$. The commutation relations of the k_α's are

$$
\begin{aligned}
[k_{b_{ij}}, k_{b_{mn}}] &= \delta_{jm} k_{b_{in}} - \delta_{in} k_{b_{mj}} + \delta_{im}(k_{b_{nj}} - k_{b_{jn}}) + \delta_{jn}(k_{b_{mi}} - k_{b_{im}}) \\
[k_{b_{ij}}, k_{b_m}] &= -\delta_{im} k_{b_j} + \delta_{jm} k_{b_i} \\
[k_{b_{ij}}, k_{a_{mn}}] &= -\delta_{im} k_{a_{jn}} - \delta_{in} k_{a_{mj}} + \delta_{im} k_{a_{nj}} + \delta_{jm} k_{b_{in}} + \delta_{jn}(k_{b_{mi}} - k_{b_{im}}) \\
[k_{b_{ij}}, k_{a_m}] &= \delta_{jm} k_{a_i} - \delta_{im} k_{a_j} \\
[k_{b_{ij}}, k_{a_m}] &= -\delta_{im} k_{a_j} + \delta_{jm} k_{a_i} \\
[k_{b_i}, k_{b_j}] &= -k_{b_{ij}} + k_{b_{ji}} \\
[k_{b_i}, k_{a_{mn}}] &= -\delta_{in} k_{a_m} + \delta_{im} k_{a_n} \\
[k_{b_i}, k_{a_j}] &= -k_{a_{ij}} + k_{a_{ji}} \\
[k_{a_{ij}}, k_{a_{mn}}] &= \delta_{jm} k_{b_{in}} - \delta_{in} k_{b_{mj}} + \delta_{im}(k_{b_{nj}} - k_{b_{jn}}) + \delta_{jn}(k_{b_{mi}} - k_{b_{im}}) \\
[k_{a_{ij}}, k_{a_m}] &= -\delta_{im} k_{b_j} + \delta_{jm} k_{b_i}
\end{aligned}
\tag{A.4}
$$

By going to the new basis $\{k_b + k_a, k_b - k_a\}$, one easily recognizes the algebra $so(n) \oplus so(n)$.

A.3 Embedding of A_{n-1}

The gravitational subalgebra A_{n-1} is generated by $h_1, \cdots, h_{n-2}, \tilde{h}_{n-1}$ (Cartan generators), $e_1, \cdots, e_{n-2}, \tilde{e}_{n-1}$ (raising operators) and $f_1, \cdots, f_{n-2}, \tilde{f}_{n-1}$ (lowering operators), with $\tilde{h}_{n-1} = -h_n - h_2 - h_3 - \cdots - h_{n-1}$. The simple root $\tilde{\alpha}_{n-1}$ is connected to α_1 only, with a single link. Note that although it is a simple root for the gravitational subalgebra A_{n-1}, it is in fact the highest root of the A_{n-1} subalgebra associated with the Dynkin subdiagram $n, 2, 3, \cdots, n-1$.

B E_8 algebra

We take a basis of the Cartan subalgebra (h_i), such that

$$\begin{aligned} [f_{ij}, e_{ij}] &= h_i - h_j \\ [f_i, e_i] &= -h_i \\ \left[\tilde{f}_{ijk}, \tilde{e}_{ijk}\right] &= \tfrac{1}{3}(h_1 + h_2 + \ldots + h_8) - h_i - h_j - h_k \\ \left[\tilde{f}_{ij}, \tilde{e}_{ij}\right] &= -\tfrac{1}{3}(h_1 + h_2 + \ldots + h_8) + h_i + h_j \end{aligned} \tag{B.1}$$

$$\begin{aligned} [h_i, e_{ij}] &= e_{ij} & [h_i, f_{ij}] &= -f_{ij} \\ [h_j, e_{ij}] &= -e_{ij} & [h_j, f_{ij}] &= f_{ij} \\ [h_i, e_i] &= -e_i & [h_i, f_i] &= f_i \\ [h_i, \tilde{e}_{ijk}] &= -\tilde{e}_{ijk} & \left[h_i, \tilde{f}_{ijk}\right] &= \tilde{f}_{ijk} \\ [h_i, \tilde{e}_{jk}] &= -\tilde{e}_{jk} & \left[h_i, \tilde{f}_{jk}\right] &= \tilde{f}_{jk} \end{aligned} \tag{B.2}$$

where distinct indices are supposed to have different values. The vectors associated with the simple roots are $e_{i\,i+1}$, $\tilde{e}_{123}$.

Other non vanishing commutations relations are the following, with the same convention on indices.

$$\begin{aligned} [e_{ij}, e_{jk}] &= e_{ik} & [f_{ij}, f_{jk}] &= f_{ik} \\ [\tilde{e}_{ijk}, e_{kl}] &= e_{ijl} & \left[\tilde{f}_{ijk}, f_{kl}\right] &= f_{ijl} \\ [\tilde{e}_{ijk}, \tilde{e}_{lmn}] &= \tfrac{1}{2}\epsilon^{ijklmnpq}\tilde{e}_{pq} & \left[\tilde{f}_{ijk}, \tilde{f}_{lmn}\right] &= \tfrac{1}{2}\epsilon^{ijklmnpq}\tilde{f}_{pq} \\ [e_{ij}, \tilde{e}_{jk}] &= \tilde{e}_{ik} & \left[f_{ij}, \tilde{f}_{jk}\right] &= \tilde{f}_{ik} \\ [\tilde{e}_{ijk}, \tilde{e}_{jk}] &= e_i & \left[\tilde{f}_{ijk}, \tilde{f}_{jk}\right] &= f_i \\ [e_i, e_{ij}] &= e_j & [f_i, f_{ij}] &= f_j \end{aligned} \tag{B.3}$$

$$\begin{aligned} [f_{ij}, e_{kj}] &= e_{ki} \quad \text{if } i > k & [f_{ij}, e_{kj}] &= -f_{ik} \quad \text{if } i < k \\ [f_{ji}, e_{jk}] &= -e_{ik} \quad \text{if } i < k & [f_{ji}, e_{jk}] &= f_{ki} \quad \text{if } i > k \\ [f_{ij}, \tilde{e}_{klj}] &= e_{kli} & \left[e_{ij}, \tilde{f}_{klj}\right] &= f_{kli} \\ \left[\tilde{f}_{ijk}, \tilde{e}_{ijl}\right] &= -e_{kl} \quad \text{if } k < l & \left[\tilde{f}_{ijk}, \tilde{e}_{ijl}\right] &= f_{lk} \quad \text{if } k > l \\ [f_{ji}, \tilde{e}_{jk}] &= -\tilde{e}_{ik} & \left[e_{ji}, \tilde{f}_{jk}\right] &= -\tilde{f}_{ik} \\ \left[\tilde{f}_{ijk}, \tilde{e}_{lm}\right] &= -\tfrac{1}{3!}\epsilon^{ijklmnpq}\tilde{e}_{npq} & \left[\tilde{e}_{ijk}, \tilde{f}_{lm}\right] &= -\tfrac{1}{3!}\epsilon^{ijklmnpq}\tilde{f}_{npq} \\ [f_{ij}, e_j] &= e_i & [e_{ij}, f_j] &= f_i \\ \left[\tilde{f}_{ij}, \tilde{e}_{ik}\right] &= e_{kj} \quad \text{if } j > k & \left[\tilde{f}_{ij}, \tilde{e}_{ik}\right] &= -f_{jk} \quad \text{if } j < k \\ \left[\tilde{f}_{ijk}, e_k\right] &= -\tilde{e}_{ij} & [\tilde{e}_{ijk}, f_k] &= -\tilde{f}_{ij} \\ \left[\tilde{f}_{ij}, e_k\right] &= \tilde{e}_{ijk} & [\tilde{e}_{ij}, f_k] &= \tilde{f}_{ijk} \\ [f_i, e_j] &= -e_{ij} \quad \text{if } i < j & [f_i, e_j] &= f_{ji} \quad \text{if } i > j \end{aligned} \tag{B.4}$$

The Chevalley-Serre generators are $h_i - h_{i+1}$, $h_{123} \equiv \frac{1}{3}(h_1 + \cdots + \eta_8) - h_1 - h_2 - h_3$, e_{ii+1}, $\tilde{e}_{123}$, f_{ii+1} and $\tilde{f}_{123}$. The scalar products of the h_i's are $K(h_i, h_j) = \delta_{ij} + 1$ and the factors N_α are equal to unity.

The generators of the compact subalgebra so(16) are

$$\begin{aligned} k_{ij} &= e_{ij} + f_{ij} \quad \text{for } i < j \\ k_i &= e_i + f_i \\ \tilde{k}_{ijk} &= \tilde{e}_{ijk} + \tilde{f}_{ijk} \\ \tilde{k}_{ij} &= \tilde{e}_{ij} + \tilde{f}_{ij} \ . \end{aligned}$$

It is convenient to define $k_{ij} = -k_{ji} = -e_{ji} - f_{ji}$ for $i > j$. Their non vanishing commutators are

$$\begin{array}{rclrcl}
[k_{ij}, k_{jk}] & = & k_{ik} & \left[\tilde{k}_{ijk}, k_{kl}\right] & = & \tilde{k}_{ijl} \\
\left[\tilde{k}_{ijk}, \tilde{k}_{lmn}\right] & = & \frac{1}{2}\epsilon^{ijklmnpq}\tilde{k}_{pq} & \left[\tilde{k}_{ijk}, \tilde{k}_{ijl}\right] & = & -k_{kl} \\
\left[k_{ij}, \tilde{k}_{jk}\right] & = & \tilde{k}_{ik} & \left[\tilde{k}_{ijk}, \tilde{k}_{jk}\right] & = & k_l \\
\left[\tilde{k}_{ijk}, \tilde{k}_{lm}\right] & = & -\frac{1}{3!}\epsilon^{ijklmnpq}\tilde{k}_{npq} & \left[\tilde{k}_{ij}, \tilde{k}_{ik}\right] & = & -k_{jk} \\
[k_{ij}, k_j] & = & k_i & \left[\tilde{k}_{ijk}, k_k\right] & = & -\tilde{k}_{ij} \\
\left[\tilde{k}_{ij}, k_k\right] & = & \tilde{k}_{ijk} & [k_i, k_j] & = & -k_{ij}
\end{array} \tag{B.5}$$

where it is assumed that distinct indices have different values.

C Algebra $G_{2(2)}$

Let e_1 and e_2 be the positive Chevalley generators of G_2 corresponding to the two simple roots α_1 and α_2. The other positive generators are

$$\begin{array}{rcll}
e_3 & = & [e_2, e_1] & e_4 = [e_2, [e_2, e_1]] \\
e_5 & = & [e_2, [e_2, [e_2, e_1]]] & e_6 = [[e_2, [e_2, [e_2, e_1]]], e_1].
\end{array} \tag{C.1}$$

Their non vanishing commutation relations are,

$$\begin{array}{rcll}
[e_1, e_2] & = & -e_3 & [e_1, e_5] = -e_6 \\
[e_2, e_3] & = & e_4 & [e_2, e_4] = e_5 \\
[e_3, e_4] & = & e_6 &
\end{array} \tag{C.2}$$

The normalizing factors N_α for the simple roots are given by $N_1 = 1$ and $N_2 = 3$ since $(\alpha_1|\alpha_1) = 2$ and $(\alpha_2|\alpha_2) = \frac{2}{3}$. It follows that $N_3 = 3$, $N_4 = 12$, $N_5 = 36$ and $N_6 = 36$. We define the vectors ϵ_i in order to absorb these factors, i.e., $\epsilon_1 = e_1$, $\epsilon_2 = \frac{1}{\sqrt{3}}e_2$, $\epsilon_3 = \frac{1}{\sqrt{3}}e_3$, $\epsilon_4 = \frac{1}{2\sqrt{3}}e_4$, $\epsilon_5 = \frac{1}{6}e_5$, $\epsilon_6 = \frac{1}{6}e_6$. This implies $K(\epsilon_i, \tau(\epsilon_i)) = -1$.

We take as compact generators $k_i = \epsilon_i + \tau(\epsilon_i)$. The commutators of the compact subalgebra are

$$\begin{array}{l}
[k_1, k_2] = -k_3, \qquad [k_1, k_3] = k_2, \qquad [k_1, k_4] = 0, \\
[k_1, k_5] = -k_6, \qquad [k_1, k_6] = k_5, \qquad [k_2, k_3] = \dfrac{2}{\sqrt{3}}k_4 - k_1 \\
[k_2, k_4] = k_5 - \dfrac{2}{\sqrt{3}}k_3, \qquad [k_2, k_5] = -k_4, \qquad [k_2, k_6] = 0, \\
[k_3, k_4] = k_6 + \dfrac{2}{\sqrt{3}}k_2, \qquad [k_3, k_5] = 0, \qquad [k_3, k_6] = -k_4, \\
[k_4, k_5] = k_2, \qquad [k_4, k_6] = k_3, \qquad [k_5, k_6] = -k_1.
\end{array} \tag{C.3}$$

In the basis

$$\begin{array}{lll}
\xi_1 = \dfrac{1}{4}(3k_1 + \sqrt{3}k_4), & \xi_2 = \dfrac{1}{4}(\sqrt{3}k_2 - 3k_6), & \xi_3 = -\dfrac{1}{4}(\sqrt{3}k_3 + 3k_5) \\
X_1 = \dfrac{1}{4}(k_1 - \sqrt{3}k_4), & X_2 = \dfrac{1}{4}(\sqrt{3}k_2 + k_6), & X_3 = -\dfrac{1}{4}(\sqrt{3}k_3 - k_5),
\end{array} \tag{C.4}$$

the commutation relations read

$$[\xi_i, \xi_j] = \varepsilon_{ijk}\xi_k, \qquad [\xi_i, X_j] = 0, \qquad [X_i, X_j] = \varepsilon_{ijk}X_k \tag{C.5}$$

and reveal the $su(2) \oplus su(2)$ structure of the algebra.

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